Optimal Control of Open-Channel Flow Using Adjoint Sensitivity Analysis

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Abstract: An optimal flow control methodology based on adjoint sensitivity analysis for controlling nonlinear open channel flows with complex geometries is presented. The adjoint equations, derived from the nonlinear Saint-Venant equations, are generally capable of evaluating the time-dependent sensitivities with respect to a variety of control variables under complex flow conditions and cross-section shapes. The internal boundary conditions of the adjoint equations at a confluence (junction) derived by the variational approach make the flow control model applicable to solve optimal flow control problems in a channel network over a watershed. As a result, an optimal flow control software package has been developed, in which two basic modules, i.e., a hydrodynamic module and a bound constrained optimization module using the limited-memory quasi-Newton algorithm, are integrated. The effectiveness and applicability of this integrated optimal control tool are demonstrated thoroughly by implementing flood diversion controls in rivers, from one reach with a single or multiple floodgates (with or without constraints), to a channel network with multiple floodgates. This new optimal flow control model can be generally applied to make optimal decisions in real-time flood control and water resource management in a watershed.

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Introduction

Controlling open channel flow is a key issue to mitigate hazardous flooding in a river basin, or to deliver water in an irrigation system according to a specified demand pattern. Typical flood controls consist of operating hydraulic structures in rivers, e.g., reservoirs in streams and/or floodgates on river banks, to regulate the peak flood discharge and eventually to prevent levees from overflowing and/or breaching. In case of emergency, flood diversion through floodgates, diversion channels, or deliberate levee breach can quickly lower the flood stage to mitigate flood hazards. For instance, the Bonnet Carré spillway in the lower Mississippi River was opened to divert flood waters in 1997 [U.S. Army Corps of Engineers (USACE) 1999]. In addition to controlling hazardous floods, flow control methodology can be used for evaluating reservoir management policies that minimize sediment scour and deposition in a river, e.g., a practice in the Yazoo River Basin, Mississippi, reported by Nicklow and Mays (2000), and a larger scale operation of flushing sediment out of the Xiaolangdi Reservoir in the lower Yellow River [Yellow River Conservancy Commission (YRCC) (2004)]. Flow control in an open channel also can enhance pollutant disposal management by controlling pollutant discharge in compliance with the environmental laws and eventually protecting water quality (e.g., Piasecki and Kappel 1997a,b). Undoubtedly, optimal flow control techniques are vitally important to practice better water resources management and to achieve sustainable development of an economy and society that largely rely on limited water resources.

The basic characteristics of open channel flow in the real world are nonlinearities and temporal/spatial nonuniformities. Therefore, optimal flow control requires a forecasting model capable of predicting the nonuniform and unsteady water flow in space and time. In case of fast propagation of a flood wave from upstream, only a very short time may be available for predicting the flood flow downstream. Due to the limited time for making a flood mitigation decision, it is crucial for decision makers to have an efficient model, which can predict flows quickly and reliably for the selection of an optimal flow control decision. Generally speaking, an optimal flow control system is of real time, which can identify dynamic variations of a nonlinear unsteady flow. In the case of flood diversion, once a flood is diverted from a levee, depression waves would be generated from the diversion point traveling both upstream and downstream. Because the propagations of the waves are also nonlinear, the effect of the flood diversion on the mitigation cannot be obtained straightforwardly. Due to the flow nonlinearities, it is difficult to establish the relationship between the control action and the responses of the hydrodynamic variables (stage and discharge).

With the advent of modern control theories and numerical methodologies, adaptive control and nonlinear optimization have made the solution of nonlinear flow control possible. An effective approach to establish the relationship between control actions and system responses is to implement sensitivity analysis for the control variables. Once a certain form of objective function for the control actions is defined, the sensitivity analysis is generally used to evaluate the gradient of the objective function. In general,
there are three methods applicable to calculations of sensitivity coefficients of control variables: (1) the influence-coefficient method; (2) the sensitivity-equation method; and (3) the adjoint-equation method (Yeh 1986). The influence-coefficient method perturbs each of the variables one at a time. The sensitivity-equation method evaluates the partial derivatives of the physical variables with respect to each independent control variable in the governing equations. These two methods calculate sensitivity coefficients directly. The adjoint-equation method, however, is based on the variational principles, which connects the sensitivities with the variations of physical variables and Lagrangian multipliers through adjoint sensitivity analysis (ASA). For models that involve a large number of control variables and comparatively few responses, e.g., the optimal time series of control variables, the ASA can be performed very efficiently. For detailed discussions about the ASA, see Cacuci (1981), Zou et al. (1993), and Navon (1998). In terms of the sensitivity-equation method, Ding et al. (2004) have evaluated several algorithms for identifying distributed Manning’s roughness coefficients in shallow water flows, including the limited-memory quasi-Newton (LMQN) method. They have concluded that the LMQN algorithms can efficiently capture the objective parameters with high accuracy in the strongly nonlinear open channel flows.

Through modern nonlinear optimization theory, many flow control approaches have been established and tested over the past years, for obtaining optimal flow control with engineering applications. In the early 1990s, Kawahara and Kawasaki (1990) proposed an optimal control of discharge release from a dam gate to reduce peak discharge in a flood and minimize water stage fluctuations, in which several control cases were achieved by using the ASA and the conjugate gradient algorithm. Hooper et al. (1991) applied a nonlinear stochastic optimization method (linear quadratic Gaussian control) to multiple-reservoir operation control subject to constraints that represented system mass balance and physical or operating limitations on reservoir releases. Sanders and Katopodes (1997) presented a control of dam gate operation employing the simplified one-dimensional (1D) Saint-Venant equations, in which a rectangular cross section was considered. Atanov et al. (1998) applied the ASA to control an upstream discharge in an irrigation canal; however, the applicability of the control model was strictly limited to the flow in a single channel with a specific trapezoidal cross section. Sanders and Katopodes (1999a) further developed a control of gate openings at both the upstream and downstream directions in a water delivery canal with a rectangular channel cross section by means of the adjoint sensitivity analysis and the BFGS algorithm [named for its discoverers, Broyden, Fletcher, Golddarb, and Shanno (Nocedal and Wright 1999)]. Recently, Dulhoste et al. (2004) applied the nonlinear control theory to control water distribution with a collocation method using the 1D fully nonlinear Saint-Venant equations. (Sanders and Katopodes 1999b,c) developed active flood hazard mitigation through optimizing selective lateral flow withdrawal. They extended the control of the 1D flood flow to the two-dimensional (2D) flow, in which the 2D ASA can give more complicated sensitivity patterns in the horizontal plane. Jaffe and Sanders (2001) further discussed the effects of breaching length and timing in a deliberate levee breaching for mitigating a flood wave in a shallow water flow. Through the preceding brief survey of the state-of-the-art open channel flow control, the results confirm that optimal controls of open channel flows are possible once the sensitivities about control variables are obtained; and the ASA can be applied to a large variety of flow control problems. Nevertheless, the application of the advanced control theory is still limited to a simple geometric configuration, i.e., only a single channel with a prescribed cross section. Up to now, there has not yet been a general model of flow control that can apply to the control of flow in a natural river or channel network.

In this paper, an optimal flow control methodology based on the ASA for controlling nonlinear open channel flows with complex geometries is presented. In order to make the flow control model applicable to most realistic flows in a natural river and channel network over a watershed, an advanced numerical model called the CCHE1D model [Wu and Vieira 2002; National Center for Computational Hydroscience and Engineering (NCCHHE) 2004], which is a systematically verified and validated tool to analyze 1D open channel flows with natural flow conditions and complex cross-section geometries in a channel network, has been selected as the forecasting model in the present optimal flow control model. The adjoint equations of the CCHE1D hydrodynamic model are obtained by applying variational principles to the nonlinear Saint-Venant equations. By means of the variational approach, the internal boundary conditions of the adjoint equations at a confluence in a channel network are derived. By imposing the internal boundary conditions on every confluence, the nonlinear flow control problem in a channel network becomes well-established. Furthermore, the LMQN algorithm for bound-constrained optimization developed by Ding et al. (2004) is utilized for solving optimal flow control problems with high computing efficiency. The bound constrained optimization can take into account the limitations of control devices, e.g., the maximum discharge of a floodgate. By integrating the aforementioned two basic modules, i.e., the CCHE1D hydrodynamic module and the LMQN optimization solver, a flow control software package has been developed to study real-time flood control under natural flow conditions in rivers or watersheds. The effectiveness and applicability of this optimal flow control model are demonstrated by applying it to five basic cases of flood diversion controls in rivers, from one reach with a single or multiple floodgates (with or without constraints), to a channel network with multiple floodgates. The results obtained are all physically reasonable. Therefore, this optimal flow control software can be applied to the design of engineering measures for controlling the over-bank floods along a reach of a natural river or a channel network in a watershed or a river basin. This new flow control methodology can generally be applied to optimal decision making in real-time flood control and water resource management in a watershed.

**Governing Equations**

The 1D nonlinear Saint-Venant equations are used for simulating open channel flow in a single channel or a channel network, which consist of a continuity equation and a momentum equation, i.e.

\[
L_1 = \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = -Q = 0 \tag{1}
\]

\[
L_2 = \frac{\partial}{\partial t} \left( \frac{Q}{A} \right) + \frac{\partial}{\partial x} \left( \frac{\beta Q^2}{2A^2} \right) + g \frac{\partial Z}{\partial x} + \frac{Q|Q|}{K^2} = 0 \tag{2}
\]

where \( L_1 \) and \( L_2 \) represent the continuity equation and the momentum equation, respectively; \( x= \) channel length coordinate; \( t= \) time; \( Q=Q(x,t) \), discharge through a cross section; \( A=A(x,t) \), cross-sectional area; \( Z=Z(x,t) \), water stage; \( \beta= \) momentum cor-
In a channel network, there may be several different hydrographs where the discharge exceeds the allowable water stage. The Saint-Venant equations, Eqs. (1) and (2), can simulate open channel flows with complex cross-section shapes.

In the simulation of an unsteady channel flow during a storm period $T$, the Saint-Venant Eqs. (1) and (2) are subject to initial and boundary conditions. The simulation can be started from an initial wet channel, i.e.

$$Q|_{t=0} = Q(x,0) \quad \text{and} \quad A|_{t=0} = A(x,0), \quad x \in [0,L]$$  (3)

where $L$=downstream channel length coordinate; and $Q(x,0)$ and $A(x,0)$ indicate, respectively, the base flow discharge and the wet cross-section area of the channel. In a single channel, a hydrograph $Q(0,t)$ is imposed at the upstream inlet at $x=0$, i.e.

$$Q|_{t=0} = Q(0,t), \quad t \in [0,T]$$  (4)

In a channel network, there may be several different hydrographs imposed at inlets of the main stem and tributaries. At the downstream outlet, a time series of stage $Z(L,t)$ can be imposed by

$$Z|_{t=0} = Z(L,t), \quad t \in [0,T]$$  (5)

or a stage-discharge rating curve $\gamma(Z)$ can be used for specifying the discharge downstream, i.e.

$$Q|_{t=0} = \gamma(Z)$$  (6)

**Adjoint Sensitivity Analysis of Open-Channel Flow**

**Objective Function**

The optimization problem for finding the optimal solution of a control variable in open-channel flow is to minimize the objective function $J$, which is generally defined as an integration of a general measuring function $f$, i.e.

$$J = \int_{0}^{T} \left[ \int_{0}^{L} f(Z, Q, q, x,t) dx \right] dt$$  (7)

where $q$ is a control variable, e.g., the lateral outflow in Eq. (1), of which values need to be identified to control the flow diversion. The measuring function $f$ can be a rational function of discharge, stage, and a control variable or a set of control variables. In this study on flood control problems, the measuring function is defined as the discrepancy between the predicted stage $Z(x,t)$ and the maximum allowable water stage $Z^{w}(x)$ in river banks:

$$f = \begin{cases} \frac{W_{c}[Z(x,t) - Z^{w}(x)]}{L T} & \text{if } Z(x,t) > Z^{w}(x) \\ 0 & \text{if } Z(x,t) \leq Z^{w}(x) \end{cases}$$  (8)

where $W_{c}$=weighting factor, which is to adjust the scale of the objective function; $\delta$=Dirac delta function; and $x_{t}$=target reaches where the water stages are to be controlled. Notice that the measuring function in Eq. (8) evaluates the stage discrepancies over the target reaches only where the predicted water stages become higher than the allowable water stage $Z^{w}(x)$, and the value of $Z^{w}(x)$ could either be constant or vary along target reaches. As for a certain form of the measuring function, Atanov et al. (1998) suggested the use of the fourth power in the measuring function of Eq. (8) because it is more sensitive to the stage discrepancy than the conventional least-square form. The writers further found that the identification iteration process to minimize the fourth power function was more stable than that of the least-square.

**Adjoint Equations**

The ASA is used to establish the relationship between flow variables and adjoint variables, and then to determine the sensitivities of the control variables under the constraints of the governing equations. In order to minimize the objective function Eq. (7) and ensure that the flow variables satisfy the governing Eqs. (1) and (2), an augmented objective function $J^*$ can be constructed in terms of a Lagrangian multiplier approach, namely

$$J^* = \int_{0}^{T} \int_{0}^{L} f(Z, Q, q, x,t) dx dt + \int_{0}^{T} \left( \lambda_{A} L_{1} + \lambda_{Q} L_{2} \right) dx dt$$  (9)

where $\lambda_{A}$ and $\lambda_{Q}$ are two Lagrangian multipliers. The sensitivities and the adjoint equations are evaluated by taking the variation of $J^*$ in Eq. (9) with respect to all flow variables and control variables. Then, by using Green’s formula and the variation operator $\delta$ in a time-space domain as illustrated in Fig. 1, the first variation of the augmented objective function, $\delta J^*$, can be obtained. Omitting the tedious derivation, the final form of $\delta J^*$ is given as

$$\delta J^* = \int_{0}^{T} \int_{0}^{L} \left( \frac{\partial f}{\partial Z} \delta Z + \frac{\partial f}{\partial Q} \delta Q + \frac{\partial f}{\partial q} \delta q \right) dx dt$$

$$- \int_{0}^{T} \int_{0}^{L} \left( \frac{\partial \lambda_{A}}{\partial t} + g \frac{\partial \lambda_{Q}}{\partial z} - \frac{Q}{A} \frac{\partial \lambda_{Q}}{\partial t} - \frac{\beta Q^{2}}{A^{3}} \frac{\partial \lambda_{Q}}{\partial x} \right) \delta Q dx dt$$

$$+ \int_{0}^{T} \int_{0}^{L} \left( \frac{2g(1+\lambda_{A})Q}{AK^{2}} \right) \delta A dx dt$$

$$\frac{\partial \lambda_{A}}{\partial x} \frac{2gQ}{K^{2}} \lambda_{Q} \delta Q dx dt + \int_{0}^{T} \int_{0}^{L} \left( \frac{\partial \lambda_{Q}}{\partial A} \frac{\lambda_{A}}{A} \delta A + \frac{\lambda_{A}Q}{A^{2}} \frac{\partial \lambda_{Q}}{\partial x} \delta Q \right) dx dt$$

$$- \frac{\lambda_{A}^{2}}{A} \delta Q dx dt + \int_{0}^{T} \int_{0}^{L} \left( \frac{2gQ}{K^{2}} \lambda_{Q} \right) \delta A dx dt$$

$$- \frac{\lambda_{A}}{A} \delta \lambda_{A} dx$$

$$\delta A$$

$$= \begin{cases} \frac{W_{c}[Z(x,t) - Z^{w}(x)]}{L T} & \text{if } Z(x,t) > Z^{w}(x) \\ 0 & \text{if } Z(x,t) \leq Z^{w}(x) \end{cases}$$  (10)

where $\delta A$, $\delta Q$, $\delta q$, and $\delta \lambda$=variations of cross-sectional area, discharge, lateral outflow, and Manning’s $n$, respectively;
$B^*=\text{channel width on the water surface}$, and therefore, $\partial A/\partial Z=B^*$; and $\Lambda=2/3\cdot 4R/(3B^*)$. Note that the contour integral in Eq. (10) is integrated along the closed boundary line $ABCD$ in the solution domain shown in Fig. 1. To find the minimum of the function $J$, all terms multiplied by the variations $\delta A$ and $\delta Q$ must be set to zero, respectively, in order to establish the optimization conditions, i.e., the adjoint equations on the two Lagrangian multipliers. Collecting these terms in regard to $\delta A$ and $\delta Q$ in Eq. (10), and setting the sum of these terms to zero, the following two adjoint equations containing the adjoint variables, $\lambda_A$ and $\lambda_Q$, are obtained:

$$\frac{\partial \lambda_A}{\partial t} + \frac{g}{B^*} \frac{\partial \lambda_Q}{\partial x} - \frac{Q}{A^2} \frac{\partial \lambda_Q}{\partial t} - \frac{\beta Q^2}{A^3} \frac{\partial \lambda_Q}{\partial x} + \frac{2g(1+\Lambda)Q|Q|}{AR^2} \lambda_Q = \frac{1}{B^*} \frac{\partial f}{\partial Z}$$

(11)

$$\frac{1}{A} \frac{\partial \lambda_Q}{\partial t} + \frac{\beta Q}{A^2} \frac{\partial \lambda_Q}{\partial x} + \frac{\partial \lambda_A}{\partial x} = \frac{2g|Q|}{K^2} \lambda_Q = \frac{\partial f}{\partial Q}$$

(12)

Multiplying Eq. (12) by $Q/A$, adding to Eq. (11), and expanding the derivative $\partial f/\partial Z$ by considering the measuring function $f$ in Eq. (8), the first adjoint equation can be updated as follows:

$$\frac{\partial \lambda_A}{\partial t} + \frac{Q}{A} \frac{\partial \lambda_A}{\partial x} + \frac{g}{B^*} \frac{\partial \lambda_Q}{\partial x} - \frac{Q}{A^2} \frac{\partial \lambda_Q}{\partial t} + \frac{\beta Q^2}{A^3} \frac{\partial \lambda_Q}{\partial x} + \frac{2g(1+\Lambda)Q|Q|}{AR^2} \lambda_Q = \frac{1}{B^*} \frac{\partial f}{\partial Z}$$

(13)

$$= \begin{cases} \frac{4W}{B^*L}[Z(x,t)-Z^{bij}(x)]^3 \delta(x-x_0), & \text{if } Z(x_0) > Z^{bij}(x_0) \\ 0, & \text{if } Z(x_0) \leq Z^{bij}(x_0) \end{cases}$$

(14)

As a result, the adjoint equations of the nonlinear Saint-Venant Eqs. (1) and (2) are composed of Eqs. (13) and (14). The adjoint equations are general formulations for the inverse analysis of 1D unsteady channel flows with arbitrary cross-section shapes. Using a similar approach, Atanov et al. (1998) obtained two adjoint equations similar to the present ones. Unfortunately, their approach is limited to a prismatic channel with a trapezoidal cross section.

**Boundary Conditions of Adjoint Equations for Single Channel**

The boundary conditions and the transversality conditions (or the final conditions) of the adjoint equations can be also sought in terms of the variation of the augmented objective function. Taking into account the contour integral in Eq. (10), which is integrated in a rectangular time-space solution domain as shown in Fig. 1, let $\delta I$ represent this contour integral; then this term also needs to be zero so as to satisfy the extreme condition of the objective function $J^*$, namely

$$\delta I = \oint \left[ \left( -\lambda_A + \frac{Q}{A^2} \lambda_Q \right) \frac{\delta A}{A} - \lambda_Q \frac{\delta Q}{A} \right] dx + \left[ \left( \frac{g}{B^*} - \frac{\beta Q^2}{A^3} \right) \lambda_Q \frac{\delta A}{A} \right. \right.$$

$$\left. + \left( \lambda_A + \frac{\beta Q}{A^2} \lambda_Q \right) \frac{\delta Q}{A} \right] dt = \int_{AB} \left( \left( \lambda_A + \frac{Q}{A^2} \lambda_Q \right) \delta A - \lambda_Q \delta Q \right) dx$$

$$+ \int_{BC} \left( \lambda_A + \frac{\beta Q}{A^2} \lambda_Q \right) \frac{\delta Q}{A} dt + \int_{CD} \left( \left( \lambda_A + \frac{Q}{A^2} \lambda_Q \right) \delta A - \lambda_Q \delta Q \right) dx$$

$$+ \int_{DA} \left( \right) = \int_{0}^{L} \left[ \left( -\lambda_A + \frac{Q}{A^2} \lambda_Q \right) \delta A \right. \left. + \left( \lambda_A + \frac{\beta Q}{A^2} \lambda_Q \right) \frac{\delta Q}{A} \right] \bigg|_{x=L} dx$$

$$- \int_{0}^{L} \left[ \left( -\lambda_A + \frac{Q}{A^2} \lambda_Q \right) \delta A \right. \left. + \left( \lambda_A + \frac{\beta Q}{A^2} \lambda_Q \right) \frac{\delta Q}{A} \right] \bigg|_{x=0} dx$$

$$- \int_{0}^{T} \left( \left( \frac{g}{B^*} - \frac{\beta Q^2}{A^3} \right) \lambda_Q \delta A \right. \left. + \left( \lambda_A + \frac{\beta Q}{A^2} \lambda_Q \right) \frac{\delta Q}{A} \right] \bigg|_{t=0} dt = 0$$

(15)

where $(\cdot)$ stands for the integrand of the contour integral. First, considering the initial conditions, Eq. (3), as specified conditions, the variations of cross-section area $A$ and discharge $Q$ at the initial time must be zero. Therefore, the integral $\int_{AB}(\cdot)$ in Eq. (15) along the line $AB (t=0)$ automatically vanishes. No initial conditions for the Lagrangian multipliers, $\lambda_A$ and $\lambda_Q$, thus exist. Secondly, because the values of $Q$ and $A$ at the final time $T$ cannot be specified, i.e., $\delta A(x,T) \neq 0$ and $\delta Q(x,T) \neq 0$, from the integral $\int_{CD}(\cdot)$ in Eq. (15) along the line $CD (t=T)$, the transversality conditions (or the final conditions) of the Lagrangian multipliers can be found:

$$\lambda_Q(x,T) = 0 \text{ and } \lambda_A(x,T) = 0, \quad x \in [0,L]$$

(16)

Due to these conditions, the adjoint Eqs. (13) and (14) must be solved backward in time. It is therefore inevitable to store the computed flow variables obtained in the forward computation of a flow over all time steps before implementation of the backward computation of the adjoint variables. Thirdly, taking the first variation of the upstream boundary condition Eq. (4), then $\delta Q(0,t)=0$ and $\delta A(0,t) \neq 0$. From the fourth integral $\int_{DA}(\cdot)$ in Eq. (15), the boundary condition of $\lambda_Q$ upstream is obtained as follows:

$$\lambda_Q(0,t) = 0, \quad t \in [0,T]$$

(17)

This means that the Lagrangian multiplier $\lambda_Q$ is always zero at the inlet. Finally, because of the downstream boundary condition Eq. (5), $\delta Z(L,t)=0$ can be derived. Using the relation $\delta A = B^* \delta Z$, and setting the second integral $\int_{BC}(\cdot)$ in Eq. (15) to zero, the boundary condition of the Lagrangian multipliers at a downstream outlet can be derived as

$$\lambda_A(L,t) = - \frac{B(L,t)Q(L,t)}{A(L,t)^2} \lambda_Q(L,t), \quad t \in [0,T]$$

(18)

**Internal Conditions for Confluence in Channel Network**

In order to solve the adjoint equations for optimal control in a channel network, it is indispensable to impose the internal boundary conditions at every confluence in a channel network. These boundary conditions at a confluence can be found by a similar variational approach as used earlier in this paper. For simplicity, the mathematical analyses regarding the conditions are briefly described as follows: Taking into account a confluence in a chan-
nel network as shown in Fig. 2, which comprises three channels, i.e., Channel 1 and Channel 2 at the upstream, and Channel 3 at the downstream, the internal boundary conditions of stages and discharges are specified as follows:

\[ Z_1 = Z_2 = Z_3 \quad \text{and} \quad Q_3 = Q_1 + Q_2 \quad (19) \]

where \( Z_1, Z_2, \) and \( Z_3 \) represent the stages in the cross sections 1, 2, and 3, respectively; and \( Q_1, Q_2, \) and \( Q_3 \) are the discharges at the three cross sections. Taking the variations of the internal boundary conditions, Eq. (19), at the confluence, and noting that \( \delta A/\delta Z=B^* \), one can readily derive the following relationships among the variations of stages and discharges at the three sections:

\[ \frac{\delta A_1}{B_1^*} = \frac{\delta A_2}{B_2^*} = \frac{\delta A_3}{B_3^*} \quad \text{and} \quad \delta Q_3 = \delta Q_1 + \delta Q_2 \quad (20) \]

where \( B_1^*, B_2^*, \) and \( B_3^* \) are channel widths on the water surfaces at the three cross sections. By analog with the calculation of the first variation in Eq. (10), the variation of \( J' \) in a channel network can be obtained by integrating over the whole channel network including all confluences through the period of computation. As a result, the formulation of \( \delta J' \) in a channel network is similar to Eq. (10). However, several additional terms in the contour integrals contributed from these confluences in a channel network need to be included. By choosing a confluence as an example, under the demand of the extreme condition, the additional terms of the variations resulting from the confluence should also vanish in the first variation of \( J' \), namely

\[
\int_0^T \left[ \left( \frac{g}{B} - \frac{B^* Q^2}{A^3} \right) \lambda_q \delta A + \left( \lambda_A + \frac{B^* Q}{A^2} \lambda_q \right) \delta Q \right]_{x=x_1} dt + \int_0^T \left[ \left( \frac{g}{B} - \frac{B^* Q^2}{A^3} \right) \lambda_q \delta A + \left( \lambda_A + \frac{B^* Q}{A^2} \lambda_q \right) \delta Q \right]_{x=x_2} dt - \int_0^T \left[ \left( \frac{g}{B} - \frac{B^* Q^2}{A^3} \right) \lambda_q \delta A + \left( \lambda_A + \frac{B^* Q}{A^2} \lambda_q \right) \delta Q \right]_{x=x_3} dt = 0
\]

(21)

where \( x_1, x_2, \) and \( x_3 \) are locations of the three cross sections at the confluence illustrated in Fig. 2. Note that \( x_1 \) and \( x_2 \) are located, respectively, at the downstream end of Channels 1 and 2, and \( x_3 \) is the upstream inlet of Channel 3; the sign of the third term in Eq. (21) therefore is negative, in contrast to the other two terms. By substituting Eq. (20) into Eq. (21), the following equations with respect to the Lagrangian multipliers are derived:

\[
\left( \lambda_A + \frac{B^* Q}{A^2} \lambda_q \right)_{x=x_1} = \left( \lambda_A + \frac{B^* Q}{A^2} \lambda_q \right)_{x=x_2} = \left( \lambda_A + \frac{B^* Q}{A^2} \lambda_q \right)_{x=x_3}
\]

(22)

Eqs. (22) and (23) are generally the internal boundary conditions of the two Lagrangian multipliers at a confluence. By imposing all internal boundary conditions on every confluence in the channel network, the adjoint equations and their boundary conditions on the channel network flows are well-defined and, therefore, are ready for numerical solution.

**Sensitivities of Flow Control Variables**

Similarly, the sensitivities of control variables for open-channel flow control can be also obtained from the variation of the objective function in Eq. (10). In flood diversion control, the lateral outflow \( q(x, t) \) at a location of diversion \( x \) is apparently an available control variable. For simplicity, this study did not consider the kind of facility installed at the place of diversion; there might be a pump station or an operational floodgate that is able to divert the desired amount of the lateral outflow. Therefore, the optimal control of the flood diversion is to minimize the objective function Eq. (7) with the measuring function Eq. (8) so as to find an optimal hydrograph of the lateral outflow at the location of diversion. To do so, the stages over target reaches can be secured from overflowing. From Eq. (10), the variation of the objective function \( J \) with respect to the control variable \( q \) at the location \( x \), was obtained as

\[
\delta J(x, t) = \int_0^T \left( \frac{\partial J}{\partial q} \right)_{x=x} \delta q(x, t) dt
\]

(24)

Because the measuring function \( f \) in Eq. (8) is independent of \( q \), the sensitivity of the lateral outflow at the flood diversion structure and a time \( t_n \) is written readily as

\[
\frac{\partial J}{\partial q} \bigg|_{x=x_n} = -\lambda_A(x_n, t_n)
\]

(25)

As a result, the Lagrangian multiplier \( \lambda_A \) determines sufficiently the sensitivity of the lateral outflow \( q \). In addition, several other control variables are also available for open-channel flow controls. These control actions could also be implemented by upstream discharge, downstream water stage, and bed friction, either individually or combined. As byproducts from the ASA, the corresponding sensitivity formulations are provided as follows: First, in terms of the variational analysis about the contour integral in Eq. (15), it has been found that the sensitivity of upstream discharge \( Q(0, t) \) has a form similar to the sensitivity of the lateral outflow in Eq. (24) because of its physical meaning being equivalent to \( q \) at the inlet. Next, the sensitivity of the downstream cross-section area or water stage can be described as follows:

\[
\delta J(A(L, t)) = \int_0^T \left( \frac{\partial f}{\partial A} \right)_{x=x} \left[ \frac{g}{B} - \frac{B^* Q^2}{A^3} \lambda_q \right]_{x=x} \delta A(L, t) dt
\]

(26)

Finally, the bed friction can also be a control action in open-channel flow control, of which the sensitivity is written as

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**Fig. 2. Configuration of confluence**

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\[ \delta J(n) = \int_0^T \int_0^L \left( \frac{\partial f}{\partial n} + \frac{2gQ}{nK^2} \right) \delta n \, dx \, dt \]  

(27)

The formulation of the sensitivity to Manning’s n can be immediately used for identification of the roughness coefficient (e.g., Ding and Wang 2005). By means of the relationships between the sensitivities of the control variables and the adjoint variables, the temporal and spatial variations of the sensitivities can be readily determined after the two adjoint equations are solved. Therefore, theoretically, it is possible to integrate a variety of control scenarios into a general model for multiple control actions in open-channel flows. However, in the following sections, only the flood diversion control is used as an example of the control actions to demonstrate the effectiveness of the present optimal approach.

### Numerical Approaches

The Saint-Venant equations, Eqs. (1) and (2), were discretized by means of the implicit four-point finite-difference scheme proposed by Preissmann (1960), which has been widely applied to simulate 1D unsteady flows. Preissmann’s scheme provides a set of interpolation functions in time and space for evaluating values of a function \( \phi \) and its derivatives with respect to time and space near the \( n \)th spatial step and the \( m \)th time step, i.e.,

\[
\phi(x,t) = \theta [\phi_{i+1} + (1-\theta) \phi_i] + (1+\theta) [\phi_{i+1} + (1-\theta) \phi_i] 
\]

(28)

\[
\frac{\partial \phi(x,t)}{\partial t} = \phi_{i+1} - \phi_i + (1-\psi) \phi_{i+1} - \phi_i 
\]

(29)

\[
\frac{\partial \phi(x,t)}{\partial x} = \phi_{i+1} - \phi_{i-1} + (1-\theta) \phi_{i+1} - \phi_i 
\]

(30)

where \( \Delta t \) = time increment; \( \Delta x \) = spatial length; \( x \in (i\Delta x, (i+1)\Delta x) \) and \( t \in (n\Delta t, (n+1)\Delta t) \); \( \theta \) and \( \psi \) are two weighting parameters less than or equal to one; and \( \phi_i = \) value of function \( \phi \) at the \( m \)th time step and the \( n \)th spatial step. The spatial length \( \Delta x \) is not limited to be constant; it can vary with reaches. For details of the numerical discretization for the Saint-Venant equations, one may refer to Wu and Vieira (2002) or Liggett and Cunge (1975). This implicit scheme has been also employed for solving the strongly nonlinear adjoint Eqs. (13) and (14), in which the value of \( \theta \) is about 0.50 in the following test cases for flood controls. Finally, a set of linear algebraic equations in regard to \( \lambda_\alpha \) and \( \lambda_\beta \) with a tridiagonal matrix are solved by using a double sweep algorithm.

### Optimization Procedure

In order to find the optimal solution of a control variable, an iterative minimization procedure is generally required due to the nonlinearity of this optimization problem. Moreover, because the values of the time-dependent control variable at every time step have to be identified, in case of only one control device, the length of the array storing the control variable is identical to the total number of time steps in a control period \( T \); this requires that a minimization procedure is able to handle a large-scale optimization problem efficiently. Ding et al. (2004) have comprehensively compared several minimization procedures and found that those procedures based on the LMQN method have the advantages of higher convergence rate, numerical stability, and computational efficiency. Among those algorithms, the limited-memory Broyden, Fletcher, Goldfarb, and Shanno (L-BFGS) algorithm is capable of optimization in large-scale problems because of its modest storage requirements using a sparse approximation to the inverse Hessian matrix of an objective function (Nocedal and Wright 1999). Fig. 3 describes a flowchart for finding the optimal solution of the control variable \( q \) used in the present study of flood diversion control. The CCHE1D hydrodynamic model (Wu and Vieira 2002) serves the purpose of computing discharge and stage. The ASA modules are to compute the Lagrangian multipliers through the adjoint Eqs. (13) and (14). The sensitivity formulation, Eq. (25), is to calculate the gradient of the objective function. The linear search approach is to determine the searching direction \( d \) and the step size \( \alpha \). The upstream hydrograph and the cross-section geometries will be the input data for the hydrodynamic model. The initially estimated lateral outflow rate can be set as a small negative value.

Due to the physical meanings of control variables and the limitations of their values known from control devices and engineering experience, e.g., capacity of discharge at a spillway or a pump station for flood diversion, the identification of the values with bound constraints about the capacities needs to be considered. There are several kinds of nonlinear optimization methods available for constrained optimization, e.g., penalty, barrier, augmented Lagrangian methods, sequential quadratic programming (SQP), etc. (Nocedal and Wright 1999). Although some optimization methods (e.g., SQP) can solve more complicated constrained problems with equality and/or inequality constraints, there are still limited algorithms capable of solving a large-scale optimization problem such as needed in the present study. L-BFGS-B is an extension of the limited memory algorithm L-BFGS. It is, at present, the only limited quasi-Newton algorithm capable of handling both large-scale optimization problems and bounds on the control variables. For further descriptions of the L-BFGS and the L-BFGS-B algorithms, one may refer to Liu et al.
and Nocedal (1989) and Byrd et al. (1995). In order to preserve the physical meaning of the control variable in every iteration step, the L-BFGS-B procedure validated by Ding et al. (2004) is thus adopted in this study. The L-BFGS-B algorithm minimizes the objective function $J$ in Eq. (7) subject to the simple bound constraints, i.e., $q_{\text{min}} \leq q \leq q_{\text{max}}$, where $q_{\text{min}}$ and $q_{\text{max}}$ mean, respectively, the maximum and minimum lateral outflow rates in a flood diversion structure.

Examples for Optimal Flood Diversion Control

The proposed optimal control methodology is validated by optimizing the lateral outflow rates in different types of flood diversions operated in both a hypothetical single channel and a channel network. The real-time and adaptive flood controls described in the paper are to control the water stages over the flooding area under the maximum allowable stages by opening one or several floodgates according to the obtained optimal flood diversion rates.

Optimal Controls of Flood Diversions in Single Channel

Case 1: One Flood Diversion Gate

In this case, a 10-km-long single channel with flat bed as shown in Fig. 4 conveyed a hypothetical flood represented by a triangular hydrograph imposed at the upstream inlet over a flood duration $T$. The triangular hydrograph at the inlet ($x=0$) is written as a time series of discharge, i.e.

\[ q(t) = \begin{cases} \frac{Q_b + \frac{Q_p - Q_b}{T_p}}{T_p}, & t \in [0,T_p) \\ \frac{Q_b + \frac{Q_p - Q_b}{T_p}}{T_p}(T-t), & t \in [T_p,T] \end{cases} \]

in which $Q_p$=peak discharge; $Q_b$=base flow discharge; and $T_p$=time to peak. The input numerical parameters in Table 1 and the parametric values of the hydrograph in Table 2 were used to run the hydrodynamic model and then solve the adjoint equations. The same time step $\Delta t$ and spatial increment $\Delta x$ were used for solving the Saint-Venant equations and the adjoint equations. The number for the quasi-Newton update, $m$, in the L-BFGS-B procedure was set as 7. This channel was assumed to have a compound cross section with a main channel and two flood plains as shown in Fig. 4. The maximum allowable stage in the entire channel was hypothesized as a constant elevation of 3.5 m, i.e., 1.5 m higher than the elevation of the flood plain. The weighting factor $W_z$ was set to a large number, $10^5$, for all the following cases, because it was found that the magnitude of the adjoint variables was very small, and scaled with $W_z$. This large number of $W_z$ can stabilize the computations of the adjoint equations and eliminate the truncation error of the computations.

The floodgate for operating the flood diversion was supposed to be located at 7 km downstream from the inlet as shown in Fig. 4, where the optimal flood diversion rate $q(t)$ was identified in order to keep the water stage below the maximum allowable one over a target reach. The target reach $x_0$ for mitigating the hazardous flood in this case was assumed to cover the whole channel. The minimization of the objective function defined in Eq. (7) with the measuring function Eq. (8) resulted in the optimal lateral outflow rate. Through the L-BFGS-B algorithm, the optimal lateral outflow was found iteratively, and the process for searching is shown in Fig. 5. Notice that the identified discharge $q(t)$ was a vector with the number of components equal to the number of

![Fig. 4. Floodgate diversion control in single channel and cross-section shape (Case 1)](image)

![Fig. 5. Iterations for searching optimal lateral outflow in Case 1](image)

<p>| Table 1. Parameters for Simulation of Channel Flow |</p>
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unit</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>km</td>
<td>10.0</td>
</tr>
<tr>
<td>$\Delta x$</td>
<td>km</td>
<td>0.5</td>
</tr>
<tr>
<td>$\Delta t$</td>
<td>min</td>
<td>5.0</td>
</tr>
<tr>
<td>$\theta$</td>
<td>—</td>
<td>$1.0^\alpha$</td>
</tr>
<tr>
<td>$\psi$</td>
<td>—</td>
<td>0.5</td>
</tr>
<tr>
<td>$n$</td>
<td>s/m$^{1/3}$</td>
<td>0.03</td>
</tr>
<tr>
<td>$m$</td>
<td>—</td>
<td>7</td>
</tr>
</tbody>
</table>

$^\alpha \theta=0.55$ was used for solving adjoint equations.

<p>| Table 2. Hydrograph Parameters, Maximum Stage $Z^{\text{obj}}$, and Weighting Factor $W_z$ |</p>
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unit</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_p$</td>
<td>m$^3$/s</td>
<td>100.0</td>
</tr>
<tr>
<td>$Q_b$</td>
<td>m$^3$/s</td>
<td>10.0</td>
</tr>
<tr>
<td>$T_p$</td>
<td>h</td>
<td>16.0</td>
</tr>
<tr>
<td>$T$</td>
<td>h</td>
<td>48.0</td>
</tr>
<tr>
<td>$Z^{\text{obj}}$</td>
<td>m</td>
<td>3.5</td>
</tr>
<tr>
<td>$W_z$</td>
<td>—</td>
<td>$10^5$</td>
</tr>
</tbody>
</table>
The first iteration of the minimization process is shown in Fig. 7. The optimal solution with only 15 s of CPU time. The objective function values and the norm of its gradient was implemented. Fig. 6 shows the iterative histories of the objective function values and the norm of its gradient (i.e., $||\nabla J(q)||_2$). One sees that the control variable converged after 15 iterations using the L-BFGS-B. The discharge obtained after 15 iterations is considered as the optimal solution for which the objective function and its gradient are minimized. Moreover, the flood control model took only about 1 s of CPU time for performing one iteration of the optimization by using a personal computer with a 330 MHz CPU. For this kind of optimization problem with 21 nodal points along the channel and 577 control parameters (577 time steps for the hydrograph), the control model found the optimal solution with only 15 s of CPU time.

As for the sensitivity of the lateral outflow described in Eq. (25), the variation of the sensitivity $\partial J/\partial q$ in space and time at the first iteration of the minimization process is shown in Fig. 7. The sensitivity to the lateral outflow characterizes consistently the flood process in the channel and reveals that the best location for the floodgate would be located as far upstream as possible. Fig. 8 shows the temporal variations of the sensitivity with respect to the $q$ at the floodgate. Because the values of the sensitivity has vanished (very close to zero after 15 iterations), the minimum condition $\partial J/\partial q=0$ has been satisfied numerically with sufficiently high accuracy, thus mathematically proving that the obtained lateral outflow discharge after 15 iterations in Fig. 5 was optimal. Comparisons of the stages in space and time between the uncontrolled flood ($q=0$) and the optimal control flood are shown in Figs. 9(a and b), respectively. In the uncontrolled flood, the highest water stage reached 4.4 m so that the whole channel would overtop the banks when the peak discharge was passing. Conversely, in the case in which the flood water was under the optimal flood diversion control, as shown in Fig. 9(b), the flood was mitigated significantly. Except for the first 2 km reach downstream from the inlet, the flood water stages in the channel were below 3.5 m. Furthermore, the spatial and temporal variation of the controlled discharge is shown in Fig. 10. The optimally controlled discharge altered the single peak discharge in the uncontrolled flood into two small peaks in the flood duration. Apparently, this flood mitigation was well implemented due to the peak flood stage reduction by the optimal flood diversion.

In order to ascertain the convergence of the present optimization approach, refinement of the spatial lengths was performed in identifying the optimal lateral outflow rate. In addition to the aforementioned spatial length ($\Delta x=500$ m) with 20 reaches, three other spatial lengths, i.e., 250, 166.67, and 125 m, were tested, by which the whole channel was refined uniformly into 40, 60, and 80 reaches, respectively. The four optimal lateral outflow rates obtained for the four different spatial lengths are compared in Fig. 11. It implies that the convergence of the proposed optimization method is excellent; even the coarsest grid size (500 m) gave a sufficiently accurate result. Therefore, this coarsest grid size was further used in the following two cases, Cases 2 and 3.

**Case 2: One Flood Diversion Gate with Constraint of Maximum Discharge**

In the previous case, without any consideration of limitations to the diversion discharge, the optimal diversion hydrograph in the designated floodgate in Fig. 5 obtained a peak of about 100 m$^3$/s. However, in realistic flood diversion structures, the maximum discharge of flood diversion may be limited. In this case, all model
parameters including channel bathymetry, inlet hydrograph, and numerical model parameters were preserved from Case 1 where the spatial length was 500 m. However, only a constraint of the maximum lateral outflow discharge was imposed on the optimization, which was to take account of the limitation of maximum discharge capacity in a real floodgate or a spillway. Without loss of generality, this constraint simply assumed that the maximum lateral outflow rate \( q_{\text{max}} \) was 50 m\(^3\)/s. The result of the constrained optimization problem using the L-BFGS-B is shown in Fig. 12. The optimal solution of the hydrograph was found after 40 L-BFGS-B iterations, a number larger than that in the previous case. By comparing this with the unconstrained optimization result (\( \Delta x = 500 \) m) in Case 1 (short dashed lines shown in Fig. 12), the maximum discharge of 50 m\(^3\)/s for the constrained diversion in Case 2 was forced to last about 15 h in the flood mitigation.

**Case 3: Three Flood Diversion Gates**

A realistic flood diversion system in a river may have several floodgates in different river reaches. For simplicity, as shown in Fig. 13, it was assumed that three floodgates were located at 2, 4.5, and 7 km. All the model parameters were the same as those in Case 1 as listed in Tables 1 and 2, while the spatial length was 500 m. By using the L-BFGS-B, three vectors of lateral outflow rates consisting of a total of 1,731 components (577 time steps for each outflow hydrograph) have been identified, and these are shown in Fig. 14. The three optimal diversion rates at three diversion gates are different. The first floodgate (furthest) is required to divert more flood water than the others at its down-
Case 4: Flood Diversion with Varying Allowable Stages

The maximum allowable stage $Z_{\text{max}}(x)$ might vary along the channel, either continuously or discontinuously, due to the variation of the heights of levees in a river. In order to demonstrate the capability of the optimal methodology in handling complex problems, this case assumed that the maximum allowable stage varied along the channel as a linear function of the channel length, i.e., $Z_{\text{max}}(x) = 5.5 + 0.00002x$, by which the allowable stages were changed from 5.5 m at the inlet to 3.5 m at the outlet. To compare the results with that in Case 1, no constraint of the allowable stage was imposed, and only one floodgate in the channel was operated hypothetically in this case. The other model parameters in this case were similar to those in Case 1. Through approximately 15 iterations of the searching process by the L-BFGS-B, the optimal lateral outflow rate shown in Fig. 17 was obtained. The spatial and temporal variation of stage and the allowable stage are shown in Fig. 18. By comparison with Case 1, it was found that the diversion rate in this case was reduced dramatically.

Case 5: Optimal Control of Flood Diversions in Channel Network

The dendritic channel network with a main stem and two branches as shown in Fig. 19 was taken into account in this case, in which the compound cross section shown in Fig. 4 was assumed in all three channels, and there were two confluences. Three triangular hydrographs as defined in Eq. (31) were imposed on the three inlets of the channels, of which the parameters are listed in Table 4. The hydrograph at the inlet of Channel 2 was assumed to be the same as that of Channel 1. The hydrograph at the inlet of Channel 3 (main stem) had a higher peak discharge than those of the other two. In the simulation of flood propagation, the channels were divided into a total of 43 short reaches with equal spatial increments ($\Delta x = 500$ m). To test the capability of the optimal control model for the channel network under complex flood diversion conditions, three floodgates were assumed to be located at three different sites of the main stem, as shown in Fig. 19. The maximum allowable stage was set to be 3.5 m over the entire channel network. The optimization of the flood diversion problem was to find the three optimal flood diversion rates, i.e., $q_1(t)$, $q_2(t)$, and $q_3(t)$, by minimizing the overflow water stage in time and space over the channel network. After about 20 iterations of the L-BFGS-B, the three optimal diversion rates were found, and these are plotted in Fig. 20. The temporal variations of the three diversion hydrographs are quite similar to those of Case 1.
3, as shown in Fig. 14. The upstream floodgate was found to be required to divert more flood water than the downstream ones due to the different importance in the floodgate locations. Finally, the temporal variations of the well-controlled stages under the optimal flood diversions at the three monitoring sites, i.e., A, B, and C as shown in Fig. 19, are depicted in Fig. 21 by comparison with the stages without flood control. The results of the controlled stages in the channel network show that the proposed optimal methodology is capable of solving flood control problems in complex watersheds.

**Conclusions**

An optimal flow control methodology based on the adjoint sensitivity analysis (ASA) for controlling nonlinear open-channel flows with complex geometries has been established and presented. The adjoint equations, derived from the nonlinear Saint-Venant equations, are generally capable of evaluating the sensitivities with respect to a variety of control variables under complex flow conditions and cross-section shapes. The internal boundary conditions at a confluence in a channel network derived by the variational approach make the flow control model applicable to solve optimal flow control problems in a channel network over a watershed.

By means of the proposed optimization approach, an optimal flow control software package has been developed, in which two basic modules, i.e., a hydrodynamic and an optimization module, were integrated. The CCHE1D hydrodynamic model developed at the National Center for Computational Hydroscience and Engineering at the University of Mississippi, a model that has been systematically verified by analytic solutions and validated by both laboratory experiments and site-specific field data of the Goodwin Creek Experimental Watershed, which has been monitored continuously for more than 20 years by the U.S. Department of Agriculture (USDA) National Sedimentation Laboratory in Oxford, Mississippi (Wu and Vieira 2002; NCCHE 2004), was adopted as the hydrodynamic module. The optimization module included a numerical solver for the adjoint equations and an optimization procedure based on the limited-memory quasi-Newton (LMQN) method developed by Ding et al. (2004), which has been proven accurate, stable, and efficient for studies of several nonlinear optimization problems (Ding et al. 2004; Ding and Wang 2005).

This integrated computational model, containing these two validated and proven powerful modules together with the special schemes to treat the boundary and internal conditions as well as to enhance the stability and efficiency, was applied to study a series...
of five basic cases of flood control in rivers, from one reach with a single or multiple floodgates (without or with constraints), to a channel network with multiple floodgates. The results obtained are all reasonable physically with high computing efficiency. This model is robust and reliable and can obtain the optimal solution of a flood control event for a channel network within minutes. Therefore, the model development phase of this project is complete.

This new optimal flow control model can be applied to the design of engineering measures for controlling over-bank floods along a reach of a natural river or a channel network in a watershed. It can also be applied to schedule effective operation of a system of flood control measures in real time during a flood propagation event. Because it is based on optimal control theory, the system of flood control measures designed as well as the operational control variable selection and processes can be obtained to achieve the objectives within the constraints prescribed.

In the near future, the research is to be extended to include selection of the total number of flood control measures, the various types of measures, and the locations of each measure to achieve the highest cost-effectiveness in terms of construction and operation while meeting all constraints. Also planned is development of a decision support system for real-time operational schedules of all flood control measures during each individual flood event for reduction or elimination of damages caused by extreme events, including terrorist actions.

Acknowledgments

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Appendix. Derivation of Eq. (10)

Because the augmented objective function $J^*$ is a multivariable function with changes in its variables $Z$, $Q$, $n$, and $q$, the first-order variation of the function, $\delta J^*$, is given by a sum of three terms:

$$\delta J^* = \int_0^T \int_0^L (\delta f + \lambda_1 \delta L_1 + \lambda_2 \delta L_2) dx dt$$

where $\delta L_1 =$ variation of the continuity Eq. (1) and $\delta L_2 =$ that of Eq. (2); the variation of the measuring function $f$ becomes
\[
\delta f = \frac{\partial f}{\partial Z} \delta A + \frac{\partial f}{\partial Q} \delta Q + \frac{\partial f}{\partial q} \delta q
\]  
(33)

where \(\delta A\) = variation of cross-section area, which has a relation with the variation of water stage, i.e., \(\delta A = B \delta Z\); \(\delta Q\) = variation of discharge; and \(\delta q\) = variation of the lateral outflow rate. The variation of Eq. (1) can be readily written as follows:

\[
\delta L_1 = \frac{\partial \delta A}{\partial t} + \frac{\partial \delta Q}{\partial x} - \delta q
\]  
(34)

The variation of the momentum equation can be derived as follows:

\[
\delta L_2 = \frac{\partial}{\partial t} \left( A \delta Q - Q \delta A \right) + \frac{\partial}{\partial x} \left( B \delta Q \delta Q - B \delta Q \delta A + g \delta A \right)
\]  
\[+ \frac{2g |Q|}{K^2} \delta Q - \frac{2g(1 + \Lambda) Q |Q|}{AK^2} \delta A + \frac{2g |Q| Q}{nK^2} \delta n\]
\(\)  
(35)

Given two arbitrary functions \(M\) and \(N\), Green’s formula reads

\[
\int_0^T \int_0^L \left( \frac{\partial M}{\partial t} + \frac{\partial N}{\partial t} \right) dx dt = \oint M dx + N dt
\]  
(36)

The temporal/spatial domain of the contour integral in Eq. (36) is defined as the closed boundary line \(ABCD\) shown in Fig. 1. Applying Green’s formula to the second integral in Eq. (32):

\[
\int_0^T \int_0^L \lambda_\delta \delta L_1 dx dt = \int_0^T \int_0^L \left( \frac{\partial (\lambda_\delta \delta A)}{\partial t} + \frac{\partial (\lambda_\delta \delta Q)}{\partial x} \right) dx dt
\]  
\[+ \int_0^T \int_0^L \lambda_\delta \delta q dx dt = \oint \left[ - \lambda_\delta \delta A dx + \lambda_\delta \delta Q dt \right]
\]  
\[+ \lambda_\delta \delta Q dx dt\]
\(\)  
(37)

The third integral in Eq. (32) can be derived by applying the same approach to Eq. (35), i.e.

\[
\int_0^T \int_0^L \lambda_\delta \delta L_2 dx dt = \oint \left[ - \left( \frac{A \delta Q - Q \delta A}{A^2} \right) \lambda_\delta dx \right.
\]  
\[+ \left( \frac{B \delta Q \delta Q - \delta Q \delta A}{A^2} + \frac{g \delta A}{B} \right) \lambda_\delta dt \]
\[+ \int_0^T \int_0^L \left( \frac{A \delta Q - Q \delta A}{A^2} \right) \frac{\partial \lambda_\delta}{\partial t} \right]
\]  
\[+ \frac{B \delta Q \delta Q - \delta Q \delta A}{A^2} + \frac{g \delta A}{B} \right) \lambda_\delta \frac{dx dt}{dx dt}
\]  
\[+ \int_0^T \int_0^L \left( \frac{2g |Q|}{K^2} \lambda_\delta \delta Q \right)
\]  
\[+ \frac{2g(1 + \Lambda) Q |Q|}{AK} \lambda_\delta \delta A + \frac{2g |Q| Q}{nK^2} \lambda_\delta \delta n \right) dx dt
\]  
(38)

Substituting Eqs. (33), (37), and (38) into Eq. (32), grouping the terms according to its variations \(\delta A, \delta Q, \delta n,\) and \(\delta q\), the first-order variation of the augmented function \(\delta J^*\) shown in Eq. (10) can be obtained.

**Notation**

The following symbols are used in this paper:

- \(A\) = cross section area;
- \(B^*\) = channel width on water surface;
- \(f\) = measuring function;
- \(g\) = gravitational acceleration;
- \(J, J^*\) = objective functions;
- \(K\) = conveyance;
- \(L\) = channel length;
- \(L_1, L_2\) = operators of continuity and momentum equations;
- \(m\) = number of quasi-Newton update in L-BFGS;
- \(n\) = Manning’s roughness coefficient;
- \(Q\) = discharge;
- \(Q_p, Q_b\) = peak and base flow discharges, respectively;
- \(Q_1, Q_2, Q_3\) = discharges at three cross sections of confluence;
- \(q, q_1, q_2, q_3\) = control variables or lateral outflow rates;
- \(q_{\text{min}}, q_{\text{max}}\) = lower and upper bounds on values of outflow rate;
- \(R\) = hydraulic radius;
- \(T\) = duration of storm or period of control action;
- \(T_p\) = time to peak discharge in hydrograph;
- \(t, t_n\) = time;
- \(W_c\) = weighting factor;
- \(x\) = channel length coordinate;
- \(x_c\) = location of flood diversion;
- \(x_0\) = target river reach;
- \(x_1, x_2, x_3\) = three cross sections at confluence;
- \(Z\) = stage;
- \(Z_{\text{allow}}\) = allowable water stage;
- \(Z_1, Z_2, Z_3\) = stage at three cross sections of confluence;
- \(\beta\) = momentum correction factor;
- \(\gamma(Z)\) = stage-discharge rating curve;
- \(\Delta t\) = time increment;
- \(\Delta x\) = spatial length;
- \(\delta\) = variation operator or Dirac delta function;
- \(\delta I\) = variation of boundary conditions;
- \(\theta\) = weighting parameter in time;
- \(\Lambda = 2/3 - 4R/(3B^*)\);
- \(\lambda_A, \lambda_Q\) = Lagrangian multipliers;
- \(\phi\) = arbitrary physical variable; and
- \(\psi\) = weighting parameter in space.

**References**


