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Optimal control of flood diversion in watershed using nonlinear optimization

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1. Introduction

Flood control to mitigate hazardous flood waters in rivers and watersheds is of vital importance for inundation prevention, flood risk management, and water resource management. By operating in-stream hydraulic structures (e.g. reservoirs, dams, floodgates, spillways, etc.), peak flood discharges and high water stages in channels during storms can be reduced so that overflow and overtopping on levees, as well as the resulting inland flooding and inundation are eventually prevented. In case of emergency when flood waters are predicted to exceed capacities of river reaches, controls of flood water diversion/withdrawal through floodgates, diversion channels, or deliberate levee breaching can quickly mitigate flood water stages over the target reaches and even the entire watershed. Among them is the optimal flood control that can give the best cost-effective flood control schedule to minimize the risk of hazardous flood waters. It also enables to find the best design of capacities and locations of flood control structures, e.g. floodgates, floodwater detention basins [5,6]. Meanwhile, optimal flood flow control methodologies can be utilized to practice best management of water resource, sediment transport (e.g. [29]), and water quality (e.g. [31]) to achieve sustainable development of economy and society that largely depend on limited water availability.

The difficulty for achieving optimal flood control in rivers and watersheds is attributed to non-uniformities of spatially-distributed flood waters, unsteadiness and nonlinearities of its dynamics. As a prerequisite, a flood simulation model must be able to predict hydrodynamic variations of nonlinear flood flows in time and space and compute efficiently and accurately flood wave propagations of in river channels. Different from man-made open channels, river cross-sections, in general, are irregular, usually, non-rectangular and/or non-prismatic. Consequently, nonlinear optimization techniques have to be developed for establishing the relationship between flood control variables/actions (i.e. lateral diversion discharge schedules in this study) and nonlinear responses of hydrodynamic variables (i.e. water stages and discharges). Among them, adjoint sensitivity analysis (ASA) based on the variational principle has been applied to find the sensitivity of hydrodynamic variables with respect to control variables in one- and two-dimensional flow models [18,33–35]. Through solving a set of adjoint equations of hydrodynamic equations, the solutions of adjoint variables can readily connect the control...
actions with nonlinear and unsteady responses of flood flows, and provide an accurate measure of sensitivity of control performance (usually defined by an objective function). Sanders and Katopodes [33] performed an ASA of flood control by operating lateral flood water outflow in a one-dimensional (1-D) straight channel. They performed an ASA of flood control by operating lateral flood actions with nonlinear and unsteady responses of flood flows, and provide an accurate measure of sensitivity of control performance, respectively.

Meanwhile, variational data assimilation (VDA) and ASA have been used for estimating unknown bathymetries of rivers and improving flood predictions by assimilating observed flow variables from measurements and satellite images (e.g. [26,21,27]). A similar variational approach for Lagrangian data assimilation in rivers applied to identification of bottom topography and initial conditions at confluences (junctions) for both the simulation model and the adjoint model, this simulation-based optimization model becomes well-integrated to be able to automatically find the optimal solutions (i.e. lateral flow diversion hydrographs) which are the governing equations in CCHE1D for simulating transport, morphodynamic processes, and water quality in river channels [39,28]. As an extensively verified and validated numerical model, CCHE1D has become a general computational software package for simulating unsteady open channel flows, sediment transport, morphodynamic processes, and water quality in river and watershed. The optimization of flood control in the paper is based on the adjoint sensitivity approach which is the way to establish the most accurate relationship between control actions and dynamic flood flows simulated by CCHE1D. The adjoint equations are derived for the nonlinear 1-D Saint–Venant equations which are the governing equations in CCHE1D for simulating open-channel flows through non-rectangular and non-prismatic cross-sections. In order to be generally applicable to find the adjoint solutions in channel network, the internal conditions of the adjoint equations at confluences (junctions) of channel network are derived on the basis of the variational analysis. By specifying internal conditions at confluences (junctions) for both the simulation model and the adjoint model, this simulation-based optimization model becomes well-integrated to be able to automatically find the optimal solutions (i.e. lateral flow diversion hydrographs) to best control flood waters occurring in all the river reaches of the channel network. Detailed numerical solution algorithms for solving the flow governing equations and the adjoint equations are developed by using an implicit time–space discretization scheme (i.e. Preissmann’s scheme). For finding the optimal solutions for the best flood control, an efficient optimization procedure based on the Limited-Memory Quasi-Newton (LMQN) method [24] is used.
In this paper, effectiveness and applicability of this integrated simulation-based optimization tool are confirmed by searching for the optimal control variables, lateral flood diversion discharge hydrographs at one or multiple floodgates to withdraw flood waters from channels of watershed. For demonstrating the optimal flood control, a hypothetical storm event causing hazardous flow in the watershed is adopted; and the channel network with non-rectangular cross sections is sufficiently complex to represent realistic watershed topography. The control objective is to best prevent potential flooding waters from overflowing or overtopping when the capacity of target reaches is predicted to be exceeded. Cost-effectiveness of different control actions is analyzed by comparing the diverted flood water volume by one floodgate control with that by multiple floodgates control. It is found that the single-floodgate control leads to an unfavorable speed-up in river flow which may create extra erosions in the channel bed; and multiple-floodgates diversion control diverts less flood waters, therefore can be a cost-effective control action. Furthermore, in order to validate the applicability of the optimization, a one-month optimal flood control is demonstrated by applying the CCHE1D-based optimization to mitigate river floods in a natural river in western Wyoming with complex river course and cross-section shapes. The following paragraphs will present mathematical formulations of the flood simulation model, the adjoint model, their internal condition, and boundary condition. By employing an implicit iterative solver, details on numerical implementation for solving both models will be addressed. Two flood control cases in a watershed in a hypothetical storm will be demonstrated. Conclusions on the study will be given finally.

2. Flood simulation model

The flood simulation model in this simulation-based optimization model is a well-established watershed model called CCHE1D [39]. It contains a geographic information system (GIS) application software to extract channel network from a digital elevation model (DEM) of a watershed (Fig. 1a), and 1-D open-channel flow model to simulate hydrodynamic variables (water stages and discharges) in watershed. The dynamic wave model in CCHE1D is adopted in the optimization model for simulating unsteady flood flows in channel network of a watershed. The governing equations of this hydrodynamic model are the 1-D nonlinear Saint–Venant equations, which consist of a continuity equation and a momentum equation:

\[ L_1 = \frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} - q = 0, \]  
\[ L_2 = \frac{\partial}{\partial t} \left( \frac{Q}{A} \right) + \frac{\partial}{\partial x} \left( \frac{\beta Q^2}{2A^2} \right) + g \frac{\partial Z}{\partial x} + g \frac{Q \frac{A}{K}}{K} = 0, \]  

where \( L_1 \) and \( L_2 \) represent the continuity equation and the momentum equation, respectively; \( x \) = channel length coordinate; \( t \) = time; \( Q = Q(x, t) \), discharge through a cross-section varying with time and space; \( A = A(x, t) \), cross-sectional area; \( Z = Z(x, t) \), water stage; \( \beta \) = momentum correction factor due to nonuniformity of velocity distribution at cross sections; \( g \) = gravitational acceleration; \( q = q(x, t) \), volumetric rate of lateral outflow (or inflow) per unit length of the channel between two adjacent cross sections, in which the lateral outflow discharge is the control variable for flood diversion in this study; the conveyance \( K = AR^{2/3}/n \), where \( n \) = Manning’s roughness coefficient, and \( R \) = hydraulic radius. The Saint–Venant equations (1) and (2) can simulate open channel flows with complex cross-section shapes.

To simulate an unsteady flood flow during a storm event, the Saint–Venant equations (1) and (2) are subject to initial and boundary conditions. The initial condition for the simulation is a base flow in a channel (i.e. the so-called “wet” channel) in which the water stages and discharges at the starting time are predetermined.

As for the upstream boundary condition, a hydrograph is usually specified for giving a time series of flood discharges through the cross-section of an upstream inlet. At downstream outlet of a channel or channel network, a time series of water stages can be imposed; or a stage-discharge rating curve can be used for specifying water stages at downstream according to the computed discharge at the downstream outlet.

As shown in Fig. 1(a), surface water flow course in watershed can be represented by a dentritic channel network. As an internal condition in channel network, a confluence is a three-way junction as shown in Fig. 1(b), which is a key element for solving the 1-D flow equations in channel network. To compute stages and discharges in a confluence, the internal conditions are assumed as follows:

\[ Z_1 = Z_2 = Z_3 \quad \text{and} \quad Q_2 = Q_1 + Q_3, \]  

where \( Z_1, Z_2, \) and \( Z_3 \) represent the stages in the cross-sections 1, 2, and 3, respectively; \( Q_1, Q_2, \) and \( Q_3 \) are the discharges at the three cross-sections. The internal conditions of a confluence defined by (3) may create reflective waves in channel network, especially when open channel flow becomes transcritical or supercritical. As CCHE1D only takes into account subcritical open channel flows, reflective waves can be negligible. However, for the treatment of non-reflective boundary conditions, one may further refer to Gejadze et al. [10,11], Sanders and Katopodes [35,36].

3. Optimization model

In this simulation-based flood control model, nonlinear optimization methodologies based on the variational principle are adopted to build a numerical optimization algorithm for searching for the optimal flood diversion discharge (hydrograph) to mitigate flood water stages in channel network of watershed. In order to make this flood control model applicable to real flows in natural river and watershed with irregular channel cross-section shapes, which are subject to hydrograph conditions of storms at upstream, and various boundary conditions in downstream, an inverse model has to be built for the dynamic wave model of CCHE1D defined by Eqs. (1) and (2). To do so, through the definition of an objective function for control flood water stages in target reaches of a watershed, the so-called “continuous” differential adjoint model based on the differential governing equations of the CCHE1D dynamic wave model is developed to establish an analytical (i.e. a differential) relationship between the control action (lateral flood diversion) and the flood states in streams determined by stages and discharges. Although, without mathematical derivations on the governing equations of simulation model, a “discrete” adjoint model can be generated by means of automatic differentiation (e.g. [13]), our preference for the

![Fig. 1. Configuration of a confluence in a channel network in watershed](image-url)
continuous adjoint model is to develop an accurate sensitivity of the objective function with respect to the control variables, so that an efficient optimization can be achieved. Moreover, a quick minimization procedure based on the Limited-Memory Quasi-Newton (LMQN) method [24] is adopted for searching for the optimal solution of the lateral flood diversion hydrograph.

3.1. Objective function

In order to find the optimal solution of flood diversion discharge \( q(x_0,t) \) at a floodgate located at \( x_0 \), the corresponding optimization problem is to minimize the objective function \( J \) that is generally defined as an integration of the discrepancy between the predicted stages \( Z(x,t) \) and the maximum allowable water stages \( Z_{0b}(x) \) in target reaches \( x_0 \) where needs to be protected, i.e.

\[
J = \int \frac{1}{L} \int_0^L W_x [Z(x,t) - Z_{0b}(x)]^4 \delta(x - x_0) \, dx \, dt \quad \text{if} \quad Z(x_0) > Z_{0b}(x_0)
\]

\[
0, \quad \text{if} \quad Z(x_0) \leq Z_{0b}(x_0),
\]

where \( L \) = channel length; \( T \) = optimal control duration; \( W_x \) = weighting factor to address importance of target river reaches and to adjust the scale of the objective function; \( \delta \) = Dirac delta function; \( x_0 \) = target reaches where the water stages need to be controlled. The value of \( Z_{0b}(x) \) could be constant or varying along target reaches. Notice that the objective function evaluates the stage discrepancies over the target reaches only where the predicted water stages exceed the allowable water stage \( Z_{0b}(x) \); therefore this is a non-differential objective function which results in a non-smooth optimization problem (please refer to [7,42], or [22] for the details).

Indeed, there should be more than one solution to minimize the function [4] if a lateral withdrawal discharge is large enough to suck most of flood waters out of the river channel, by which a supercritical flow may occur in channels. However, that is an undesirable flood control action because over-withdrawal flood waters need a huge detention basin to store them. Practically, one would like to search for the best flood diversion control by which the flood water will be withdrawn as less as possible. The optimal solution with this physical meaning/constraint for the flood diversion control is dependent on the searching direction of the minimization procedure or an appropriated initial guess of the control variable. The examples in the papers demonstrate that with a small non-zero guess of the lateral withdrawal discharge will guarantee to find out the physically-meaning-dependent optimal solutions. Detailed explanations will be presented in the following sections.

3.2. Adjoint sensitivity analysis of flood flow model

3.2.1. Adjoint equations

In general, based on the variational analysis for a nonlinear and unsteady dynamic system, the ASA is the best approach to precisely establish the quantitative relationship between dynamic control variables and adjoint variables of the complex dynamic system. As for the dynamic wave model (i.e. Eqs. (1) and (2)) for modeling spatio-temporal variations of flood flows, the flood control action in the study is to divert flood water from channels so that water stages in channels are lower than a safe elevation. In order to minimize the objective function in (4) and guarantee that the flow variables satisfy the governing Eqs. (1) and (2), an augmented objective function \( J^* \) is introduced in terms of the Lagrangian multiplier approach, namely,

\[
J^* = J + \int_0^T \int_0^L (\lambda_A L_1 + \lambda_Q L_2) \, dx \, dt,
\]

where \( \lambda_A \) and \( \lambda_Q \) are two Lagrangian multipliers associated with the two nonlinear governing equations, respectively. According to the stationary (necessary) condition of the optimal solutions, the first-order gradient of the augmented objective function should be vanished, if the optimal control action is the solution of the nonlinear dynamic system. Therefore, taking the variation of \( J^* \) with respect to all flow variables (i.e. stages and discharges) and the control variable \( q(x_0,t) \) in a time–space domain, the first-order variation of the augmented objective function can be obtained as follows:

\[
\delta J^* = \frac{\partial J}{\partial A} \delta A + \frac{\partial J}{\partial Q} \delta Q + \frac{\partial J}{\partial q} \delta q + \int_0^T \int_0^L (\lambda_A \delta L_1 + \lambda_Q \delta L_2) \, dx \, dt
\]

\[
- \int_0^T \int_0^L \left( \frac{\partial \lambda_A}{\partial t} + \frac{\partial \lambda_A}{\partial x} \frac{Q}{A} + \frac{\partial \lambda_Q}{\partial x} \frac{Q^2}{A^2} \right) \delta \lambda_A \, dx \, dt
\]

\[
+ 2g(1 + \lambda_A) Q(t_0) \frac{Q}{A} \delta A \, dx \, dt
\]

\[
- \int_0^T \int_0^L \left( \frac{1}{A} \delta \lambda_Q + \frac{\partial Q}{\partial x} \frac{\partial \lambda_A}{\partial x} + \frac{\partial \lambda_Q}{\partial x} \frac{\partial \lambda_Q}{\partial x} - 2gQ(t_0) \right) \delta Q \, dx \, dt
\]

\[
+ \int \left[ I_1 \delta A + I_2 \delta Q \right] \, dx \, dt.
\]

where \( \delta \) is the variation operator,

\[
I_1 = -\lambda_A + \frac{Q}{A} \lambda_Q,
\]

\[
I_2 = -\lambda_A \frac{Q}{A},
\]

\[
I_3 = \left( \frac{g}{B} - \frac{gQ^2}{A^3} \right) \lambda_Q,
\]

\[
I_4 = \lambda_A + \frac{Q}{A} \lambda_Q
\]

and \( J = 2/3 - 4B'(3B^*) \), \( B^* = B(x,t) \), the channel width on water surface. Similar to the derivation procedure given by Ding and Wang [5], to satisfy with the stationary condition (i.e. \( \delta J^* = 0 \)), all the terms multiplied by the two variations, \( \delta A \) and \( \delta Q \), can be set to zero, respectively, including the last contour integral associated with the boundary conditions. This contour integral is defined over a closed time–space rectangular domain, i.e. \( 0 \leq t \leq T \) and \( 0 < x < L \).

As a result, two adjoint equations which describe the variations of the two adjoint variables \( \lambda_A \) and \( \lambda_Q \) in space and time can be obtained. Through a simplification procedure, the two adjoint equations are written as follows:

\[
\frac{\partial \lambda_A}{\partial t} + V \frac{\partial \lambda_A}{\partial x} + g \left( \frac{Q}{A} \right) \lambda_Q + \frac{2gAIV}{K^2} \lambda_Q = \frac{\partial W}{\partial x}(Z(x,t) - Z_{0b}(x))^3 \delta(x - x_0), \quad \text{if} \quad Z(x_0) > Z_{0b}(x_0)
\]

\[
0, \quad \text{if} \quad Z(x_0) \leq Z_{0b}(x_0),
\]

\[
\frac{\partial \lambda_Q}{\partial t} + A \frac{\partial \lambda_Q}{\partial x} + BV \frac{\partial \lambda_Q}{\partial x} - \frac{2gAIV}{K^2} \lambda_Q = 0,
\]

where the cross-section velocity \( V = Q/A \). The adjoint equations are general formulations for the inverse analysis of 1-D unsteady channel flows with arbitrary cross-sections shapes. According to Le Diem's variational data assimilation theory [20], the above adjoint equations are categorized into the first-order adjoint equations. Eqs. (8) and (9) are also first-order partial differential equations, which can be rewritten into a compact vector form:

\[
\frac{\partial U}{\partial t} + M \frac{\partial U}{\partial x} + NU + P = 0,
\]

where \( U = \left( \begin{array}{c} \lambda_A \\ \lambda_Q \end{array} \right) \), \( M = \left( \begin{array}{cc} V & g \\ A & BV \end{array} \right) \), \( N = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \), and \( P \) represents the source term related to the objective function as shown in the right hand side of the adjoint equations. From the eigenvalues of the coefficient matrix \( M \), it is found that this adjoint model has...
two characteristic lines with the following two real and distinct
eigenvalues:
\[ \lambda_{1,2} = \frac{\beta + 1}{2} V \pm \sqrt{\frac{(\beta - 1)^2}{4} V^2 + \frac{g}{B} A} \]  
(11)

It indicates that the adjoint system (10) is linear and hyperbolic. In the case of a flow in a prismatic open channel, \( \beta = 1 \), therefore, \( \lambda_{1,2} = V \pm \sqrt{\frac{g}{B} A} \); the characteristic lines of the adjoint model are the same as those of the dynamic wave model shown in Eqs. (1) and (2) [35,3]. It can be found in the following content that due to the transversality condition of the adjoint variables, following the same characteristic lines, the travel directions of the perturbation waves of the adjoint variables are opposite to those forward time directions of flood flow waves. Gejadez et al. [10,11], also derived two characteristic lines for a set of adjoint equations associated with a vertical two-dimensional free-surface flow. Since this inverse model is strongly dependent of the governing Eqs. (1) and (2), the adjoint equations only can be solved numerically.

3.2.2. Sensitivities of flood diversion

The sensitivities of control variables (flood diversion discharge \( q(x, t) \)), which is identical to the gradient of the objective function \( J \) with respect to \( q \), also can be obtained from the first variation of the objective function. For simplicity, this study does not consider the actual type of the control facility installed at the place of diversion; there might be a pump station or an operational floodgate to the control variable (i.e. the flood diversion discharge). For simplicity, this study does not consider the actual type of the control variable (i.e. the flood diversion discharge).

By the formulation (12), the adjoint Lagrangian multiplier \( \lambda_A \) precisely determines the sensitivity of the objective function with respect to the control variable (i.e. the flood diversion discharge).

3.2.3. Transversality and boundary conditions of adjoint equations

Solving to the hydrodynamic equations, the adjoint Eqs. (8) and (9) must be computed numerically with boundary conditions and certain conditions in time. It is well known that the initial conditions for the adjoint variable at \( t = 0 \) do not exist in the adjoint equations of the inverse model. Only the final conditions at the end of control action, \( t = T \), can be given, which are called as the transversality conditions for inverse models. The transversality condition and the boundary conditions can be obtained from the control integrals as the last term of the first variation of the augmented objective function in Eq. (6).

\[ \delta J = \int_t^T \left[ I_1 \lambda_A + I_2 Q \right] \, dt \quad - \int_0^t \left[ I_1 \lambda_A + I_2 Q \right] \, dt \quad - \int_0^T \left[ I_3 \lambda_A + I_4 Q \right] \, dt \]  
(13)

Considering the initial condition of river flow, by which the discharges and water stages in channels are predetermined, therefore, \( \delta Q(x, 0) = 0 \) and \( \delta \lambda(x, 0) = 0 \). From the specified boundary condition of the discharge at an inlet of upstream, it can be derived that \( \delta Q(x, t) = 0 \), but \( \delta \lambda(x, t) \neq 0 \). Similarly, At the outlet of downstream, for specified water stages or a given stage-discharge rating curve, the conditions of the variations of discharge and water stage at the outlet turn to be \( \partial \lambda(A(L, t)) = 0 \), but \( \partial Q(L, t) \neq 0 \). Therefore, considering the definition of the integrands in Eq. (7), Eq. (13) can be extended to the following form:

\[ \delta J = \int_t^T \left[ \int_0^A \left( \frac{\beta Q}{A^2} \frac{\partial Q}{\partial A} \right) \, dx \right] \, dt - \int_0^t \left( \frac{\partial \lambda_A}{A^2} \right) \, dt \]  
(14)

To assure the stationary condition for the optimal solution of the dynamic system, the variations of the boundary condition \( \delta I \) must be zero. Consequently, from the second term in Eq. (14), the transversality conditions of the adjoint equations can be readily obtained, i.e.

\[ \lambda_Q(x, T) = 0 \quad \text{and} \quad \lambda_L(x, T) = 0, \quad x \in [0, L]. \]  
(15)

Due to the conditions given at the end of control period, the adjoint Eqs. (8) and (9) must be solved backward in time. It is therefore inevitable to store the computed flow variables obtained in the forward computation of a flow over all time steps before the backward computation of the adjoint variables is carried out. It implies that along with the characteristic lines given in (11), the perturbation waves of the adjoint variables, traveling backward in time, are opposite to the wave propagation directions of flows. Furthermore, by setting the third term of Eq. (14) to zero, the boundary condition of \( \lambda_Q \) at upstream (i.e. \( x = 0 \)) is given as follows:

\[ \lambda_Q(x, 0) = 0, \quad t \in [0, T]. \]  
(16)

This means that the Lagrangian multiplier \( \lambda_Q \) is always zero at the upstream inlet. This upstream condition of the adjoint variable can be also applied to every inlet cross-section in channel network. Finally, from the first term of \( \delta J \) in (14), the two Lagrangian multipliers at the downstream outlet at \( x = L \) have the following relationship, i.e.

\[ \lambda_L(L, t) = \frac{\beta(L, t) Q(L, t)}{A(L, t)^2} \lambda_Q(L, t), \quad t \in [0, T]. \]  
(17)

By applying the transversality condition (15), the upstream condition (16), and the downstream condition (17), the adjoint Eqs. (8) and (9) become a well-defined inverse model. It should be noted that the flow model and the adjoint model are suitable for modeling and optimizing subcritical flows or transcritical flows, but probably not supercritical flows, since the water stages at downstream are given. For treatment of non-reflective boundary conditions of the adjoint variables, one may further refer to Sanders and Katopodes [35], Sanders and Katopodes [36], Gejadez et al. [10,11].

3.2.4. Internal conditions at a confluence in channel network

In order to solve the adjoint equations for optimal control in a watershed represented by channel network, the internal conditions Eq. (3) have to be imposed at every confluence in channel network which are shown in Fig. 1(b). Similar to the variational analysis of the boundary conditions at the upstream and the downstream as discussed above, the variation of the contour integral over boundary, i.e. \( \delta I \) shown in Eq. (13), can be extended at a confluence as shown in Fig. 1(b). There are three terms associated with the three cross-sections at a confluence added to the variation, i.e.

\[ \delta J_{\text{confluence}} = \int_t^T \left[ I_3 \lambda_A + I_4 Q \right] \, dt + \int_0^T \left[ I_3 \lambda_A + I_4 Q \right] \, dt \]  
(18)

where \( x_1, x_2, x_3 \) represent the locations of two upstream cross-sections and one downstream cross-section at the confluence. By recalling the internal conditions for discharges and stages at the
confluence shown in Eq. (3), and considering \( \delta Z = \delta A/\delta t \), the relationship between the variations of the variables at the three cross sections is readily written as follows:

\[
\frac{\delta A_1}{B_1} - \frac{\delta A_2}{B_2} - \frac{\delta A_3}{B_3} = \delta Q_1 + \delta Q_2.
\]

Using this relationship, therefore, Eq. (18) is rearranged as:

\[
\frac{\delta l}{\delta t}_{\text{confluence}} = \int_0^L \left( \frac{B_3}{B_1} l_{x=1} - \frac{B_2}{B_1} l_{x=2} - \frac{B_1}{B_1} l_{x=3} \right) \delta A_1 + \left[ l_{x=1} - l_{x=3} \right] \delta Q_1 \, dt.
\]

To satisfy the stationary condition of the optimal solution, this variation on the internal boundary at a confluence should be vanished. Namely, the terms before the variations \( \delta A_1 \), \( \delta Q_1 \) and \( \delta Q_2 \) become zero, respectively. Considering the definitions of the integrands in Eq. (7), three internal conditions for the adjoint equations are obtained as follows:

\[
(\omega u)_{x=3} = (\omega u)_{x=1} + (\omega u)_{x=2},
\]

\[
\left( \lambda + \frac{\mu Q_{j1}}{A^2} \right)_{x=3} = \left( \lambda + \frac{\mu Q_{j1}}{A^2} \right)_{x=1} + \left( \lambda + \frac{\mu Q_{j1}}{A^2} \right)_{x=2},
\]

where \( u = g - \frac{\sigma^2}{\rho} \). The above Eqs. (21) and (22) are generally the internal conditions of the two Lagrangian multipliers, \( \lambda_A \) and \( \lambda_Q \), at a confluence. By imposing these internal conditions at each confluence in channel network, this inverse model governed by the adjoint equations, subject to the transversality conditions (15) and their boundary conditions (16) and (17), becomes well-posed. The adjoint equations, dependent on the flow model results, can only be solved numerically.

4. Numerical implementation for solving simulation model and adjoint equations

The Saint–Venant equations (1) and (2) in the wave dynamic model of CCHE1D were discretized by means of implicit four-point finite difference scheme proposed by Preissmann [32], which has been widely used to simulate 1-D unsteady flows. Provided that the computational domain as shown in Fig. 2 is discretized in time and space, this scheme gives a set of interpolation functions in time and space for evaluating values of a variable \( \phi \) and its derivatives with respect to time and space, i.e. \( \frac{\partial \phi}{\partial t} \) and \( \frac{\partial \phi}{\partial x} \), near the ith spatial step and the n th time step, i.e.

\[
\phi(x, t) = \theta(\phi_{i+1}^{n+1} + (1-\theta)\phi_{i+1}^{n}) + (1+\theta)(\phi_{i}^{n+1} + (1-\theta)\phi_{i}^{n}),
\]

\[
\frac{\partial \phi(x, t)}{\partial t} = \frac{\psi}{\Delta t}(\phi_{i+1}^{n+1} - \phi_{i+1}^{n}) + \frac{1-\psi}{\Delta t}(\phi_{i}^{n+1} - \phi_{i}^{n}),
\]

\[
\frac{\partial \phi(x, t)}{\partial x} = \frac{\theta}{\Delta x}(\phi_{i+1}^{n+1} - \phi_{i+1}^{n}) + \frac{1-\theta}{\Delta x}(\phi_{i}^{n+1} - \phi_{i}^{n}).
\]

where \( \Delta t = \text{time increment}; \Delta x = \text{spatial length}; x \in (i\Delta x, (i+1)\Delta x) \) and \( t \in (n\Delta t, (n+1)\Delta t) \); \( \theta \) and \( \psi \) are two weighting parameters less than or equal to one; \( \phi_t \) = the value of function \( \phi \) at the n th time step and the i th spatial step. The spatial length \( \Delta x \) is not limited to be constant; it can vary with reaches. Using Preissmann’s scheme, the discretized continuity equation and the momentum equation are numerically coupled. By assuming a linear relationship between the incremental values of discharge and stage, Liggett and Conde [23] obtained a set of linear algebraic equations with a pentadiagonal matrix which are solved by a double sweep algorithm. Wu and Vieira [39] adopted the same approach to solve the Saint–Venant equations in CCHE1D and to apply to modeling flows in channel network in watersheds and practical engineering problems with various in-stream hydraulic structures such as culverts, bridge crossing, measuring flume, etc.

This implicit scheme is also used for discretization of the adjoint Eqs. (8) and (9). The first equation for the adjoint variable \( \lambda_A \) is discretized as follows:

\[
\frac{\psi}{\Delta t}(x_{i+1}^{n+1} - x_{i}^{n+1}) + \frac{1-\psi}{\Delta t}(x_{i+1}^{n} - x_{i}^{n}),
\]

\[
+ V_{i+1}^{n+1} \frac{\theta}{\Delta x}(x_{i+1}^{n+1} - x_{i}^{n+1}) + \frac{1-\theta}{\Delta x}(x_{i+1}^{n} - x_{i}^{n}),
\]

\[
+ \frac{g}{(B')_{i+1}^{n+1} \Delta t}(x_{i+1}^{n+1} - x_{i}^{n+1}) + \frac{1-\theta}{\Delta x}(x_{i+1}^{n} - x_{i}^{n}),
\]

\[
+ 2g \left( \frac{\Delta A V}{V_{i}^{n+1} K} \right)^{n+1}_{i+1} \left[ \theta(\psi_{i+1}^{n+1} + 1 - \psi) + (1 - \theta)(\psi_{i+1}^{n} + 1 - \psi) \right],
\]

\[
= \begin{cases} 
\frac{4\nu_{i+1}^{n+1} Z(x_{i+1}^{n+1} - Z(x_{i}^{n+1}))}{i+1} 
& \text{if } Z(x_{i}^{n+1}) > Z(x_{i+1}^{n+1}) \\
0 & \text{if } Z(x_{i}^{n+1}) < Z(x_{i+1}^{n+1}) 
\end{cases}
\]

where

\[
V_{i+1}^{n+1} = \theta(\psi_{i+1}^{n+1} + 1 - \psi) + (1 - \theta)(\psi_{i+1}^{n} + 1 - \psi).
\]

The other terms in (26) such as \((B')_{i+1}^{n+1}\) and \((\Delta A V/V_{i}^{n+1})_{i+1}^{n+1}\) can be calculated using the same discretization method as shown in (27), (23). Similarly, the second adjoint equation for \( \lambda_Q \) has the following discretized form:

\[
\frac{\psi}{\Delta t}(x_{i+1}^{n+1} - x_{i}^{n+1}) + \frac{1-\psi}{\Delta t}(x_{i+1}^{n} - x_{i}^{n}),
\]

\[
+ A_{n+1}^{n+1} \frac{\theta}{\Delta x}(x_{i+1}^{n+1} - x_{i}^{n+1}) + \frac{1-\theta}{\Delta x}(x_{i+1}^{n} - x_{i}^{n}),
\]

\[
+ (\beta V_{i}^{n+1} K)_{i+1}^{n+1} \left[ \theta(\psi_{i+1}^{n+1} + 1 - \psi) + (1 - \theta)(\psi_{i+1}^{n} + 1 - \psi) \right],
\]

\[
= \begin{cases} 
\frac{4\nu_{i+1}^{n+1} Z(x_{i+1}^{n+1} - Z(x_{i}^{n+1}))}{i+1} 
& \text{if } Z(x_{i}^{n+1}) > Z(x_{i+1}^{n+1}) \\
0 & \text{if } Z(x_{i}^{n+1}) < Z(x_{i+1}^{n+1}) 
\end{cases}
\]

where \((A_{n+1}^{n+1})_{i+1}^{n+1}\) and \((\beta V_{i}^{n+1} K)_{i+1}^{n+1}\) can also be calculated using the same Preissmann scheme (23) and (27). In computation of the adjoint equations, the value of \( \psi \) is about 0.50 for flood controls, and that of \( \psi \) is set to 0.55. Note that the adjoint equations are solved backward in time, i.e. suppose that the adjoint variables at the \((n+1)\)th time step are known, two discretized equations are rearranged into the following algebraic equations:

\[
a_i x_{i+1}^{n+1} + b_i x_{i}^{n+1} + c_i x_{i+1}^{n+1} + d_i x_{i}^{n+1} + p_i = 0,
\]

\[
e_i x_{i+1}^{n+1} + f_i x_{i}^{n+1} + g_i x_{i+1}^{n+1} + r_i x_{i}^{n+1} + w_i = 0.
\]

![Fig. 2. Computational domain.](image-url)
where

\[ a_i = \frac{1 - \psi}{\Delta t} + \frac{1 - \theta}{\Delta x} S_{x,i+\delta}, \]

\[ b_i = \frac{g}{(B')_{i+\delta}} \frac{1 - \theta}{\Delta x} + 2g \left( \frac{AV_i}{K^2} \right)_{i+\delta} (1 - \theta)(1 - \psi), \]

\[ c_i = \frac{\psi}{\Delta t} - \frac{1 - \theta}{\Delta x} S_{x,i+\delta}, \]

\[ d_i = -\frac{g}{(B')_{i+\delta}} \frac{1 - \theta}{\Delta x} + 2g \left( \frac{AV_i}{K^2} \right)_{i+\delta} (1 - \theta)\psi, \]

\[ p_i = \frac{1 - \psi}{\Delta t} + \frac{1 - \theta}{\Delta x} \rho^i_{\phi}, \]

\[ q_i = -\frac{\psi}{\Delta t} - \frac{1 - \theta}{\Delta x} \rho^i_{\phi}, \]

\[ r_i = \frac{\psi}{\Delta t} - \frac{1 - \theta}{\Delta x} \rho^i_{\phi} + 2g \left( \frac{AV_i}{K^2} \right)_{i+\delta} (1 - \theta)\psi, \]

\[ w_i = \frac{\theta}{\Delta t} A^i_{\phi}, \]

\[ + \frac{1 - \psi}{\Delta t} - \frac{\rho^i_{\phi}}{\Delta x} \frac{\theta}{\Delta t} - 2g \left( \frac{AV_i}{K^2} \right)_{i+\delta} (1 - \theta)\psi. \]

The discretized adjoint Eqs. (26) and (28), are linear as the coefficients are dependent on the computed flows. Note that the adjoint equations are solved backward in time, due to the transversality condition. To solve these two equations, a double sweep algorithm is employed, which contains two recursive loops, backward and forward. For the details of the algorithm, one may refer to Liggett and Cunge [23]. Here, the formulations for the computations are given as follows: In forward computation, the following coefficients are calculated over the whole computational domain (all river reaches) in the nth time step:

\[ S_{x,i+1} = \left( \epsilon_i + f S_i \right) c_i - (a_i + b_i S_i) g_i, \]

\[ (a_i + b_i S_i) \frac{\partial}{\partial x} - (\epsilon_i + f S_i) \frac{d \partial}{d t}, \]

\[ T_{x,i+1} = \left( w_i - f T_i \right) (a_i + b_i S_i) - \left( p_i - b_i T_i \right) (a_i + f S_i), \]

\[ (a_i + b_i S_i) \frac{\partial}{\partial x} - (\epsilon_i + f S_i) \frac{d \partial}{d t}. \]

In backward computation, the adjoint variables are calculated by the following formulations:

\[ \lambda_{\phi,i}^a = \left( p_i - b_i T_i \right) - C_{\phi,i} L_{\phi,i}^a + d_{\phi,i} L_{\phi,i}^a, \]

\[ \lambda_{\phi,i}^b = S_{x,i} A_{\phi,i}^a + T_{x,i}. \]  

In forward computation, the coefficients in (31) and (32) are calculated recursively, with the index i varying from 1 to \( l_{\text{max}} \), i.e. from upstream to downstream in river reaches, where \( l_{\text{max}} \) is the total number of cross sections. The values of \( S_i \) and \( T_i \) are given by the upstream boundary conditions (16), i.e. \( S_1 = T_1 = 0 \). In the backward computation, the adjoint variables \( \lambda_A \) and \( \lambda_Q \) are computed from downstream to upstream, while the downstream condition (17) is specified to the outlet cross section.

4.1. Implementation of internal conditions at a confluence for solving adjoint equations

Similar to the implementation of the internal conditions in (3) for computation of flows in a confluence [39], the internal conditions of the two adjoint variables in (21) and (22) must be specified at every confluence of channel network, when using the double sweep algorithm to solve the adjoint equations. In general, both internal conditions for the flow model and the adjoint model may be reflective so that non-reflective boundary conditions may need to be implemented [10,11]. However, since this flow model (CCHEID) is developed for simulating flows in a dendritic watershed, it is supposed that an open channel flow in the watershed is unidirectional, i.e. from upstream to downstream, wave reflections at confluences (internal boundaries) and the outlet are negligible.

Taking into account a confluence in a dendritic channel network as shown in Fig. 1(b), in the forward computation by the double sweep algorithm, two upstream cross sections have the relationships between the two adjoint variables \( \lambda_A \) and \( \lambda_Q \) established by (34), and the coefficients \( S_3 \) and \( T_3 \), defined at the two sections have been calculated using the formulations (31) and (32). At the downstream cross-section, i.e. \( x = x_0 \), of the confluence, instead of using the two formulations, the coefficients \( S_3 \) and \( T_3 \) should be calculated accordingly by considering the internal conditions (21) and (22). The derivation process for obtaining the two coefficients at the downstream cross section is briefly explained as follows: According to the relationship in (34),

\[ \lambda_{\phi,i} = \lambda_{\phi,i+1} + \frac{T_{x,i}}{S_{x,i}}. \]

\[ \lambda_{\phi,i} = \lambda_{\phi,i+1} + \frac{T_{x,i}}{S_{x,i}}. \]

where the subscripts \( x_1 \) and \( x_0 \) indicate that the variable values are given at the two upstream cross sections, respectively. Substituting (35) and (36) into the internal condition (22), rearrange as follows:

\[ \lambda_{\phi,a} = \gamma_1 \lambda_{\phi,a} + \eta_1 \lambda_{\phi,a} + \epsilon_1, \]

\[ \lambda_{\phi,b} = \gamma_2 \lambda_{\phi,b} + \eta_2 \lambda_{\phi,b} + \epsilon_2, \]

where

\[ \gamma_1 = \frac{S_{x,i} - \lambda_{\phi,i}^{a}}{1 + S_{x,i} \left( \frac{\partial}{\partial x} \right)_{x}}, \]

\[ \eta_1 = \frac{S_{x,i} - \lambda_{\phi,i}^{a}}{1 + S_{x,i} \left( \frac{\partial}{\partial x} \right)_{x}}, \]

\[ \epsilon_1 = \frac{T_{x,i}}{1 + S_{x,i} \left( \frac{\partial}{\partial x} \right)_{x}}. \]

Substituting (37) and (38) into (21), rearrange the formulation as the same as Eq. (34), i.e. \( \lambda_{\phi,a} = S_{x,i} \lambda_{\phi,a}^{a} + T_{x,i} \), the coefficients \( S_{x,i} \) and \( T_{x,i} \) can be written as
\[ S_{x3} = \frac{u_{x1} \gamma_1 + u_{x2} \gamma_2}{u_{x3} - u_{x1} \eta_1 - u_{x2} \eta_2}, \quad (39) \]
\[ T_{x3} = \frac{u_{x1} e_1 + u_{x2} e_2}{u_{x3} - u_{x1} \eta_1 - u_{x2} \eta_2}, \quad (40) \]
where \( u_{x1}, u_{x2} \), and \( u_{x3} \) are the values of \( g - \frac{\rho q^3}{C_0} \) defined at three cross sections of a confluence, respectively. As a result, in the forward computation of the double sweep algorithm, the coefficients \( S_{x3} \) and \( T_{x3} \) for a downstream cross section of a confluence are calculated by using (39) and (40). In the backward computation, the values of \( S_{x3} \) and \( T_{x3} \) are calculated by using (33) and (34), and the adjoint variables at the two upstream cross sections are computed by using Eqs. (35)–(38).

5. Optimization algorithm

In this integrated simulation-based optimization software package, the CCHE1D hydrodynamic model is used to predict discharges and stages in channels, the adjoint model is to compute the two Lagrange multipliers through the adjoint Eqs. (8) and (9), as well as the gradient of the objective function, i.e., the sensitivity calculated from Eq. (12). The upstream hydrographs, the cross-section geometries, and the base flow discharge are the input data for the hydrodynamic model. In order to find the optimal solution of the control variable, an iterative minimization procedure is generally required due to the nonlinearity of the flow control problem. Moreover, since the values of the time-dependent control variable \( q(x, t) \) at all the time steps must be determined, in case of one floodgate for control, the length of the array storing \( q(x, t) \) is equal to the total number of time steps over a control period \( T \). This requires that a minimization procedure has to be able to efficiently handle a large-scale optimization problem when the simulation period \( T \) is long. Zou et al. [43] pointed out that the LMQN method is useful for solving large-scale optimization problems. Ding et al. [4] compared several minimization procedures based on the LMQN and a trust region approach, and found that the procedures based on the LMQN method have the advantages of higher convergence rate, numerical stability, and computational efficiency. Amongst those algorithms, the Limited-memory Broyden, Fletcher, Goldfarb, and Shanno (L-BFGS) algorithm is capable of achieving optimization in large-scale problems because of the modest storage requirements using a sparse approximation to the inverse Hessian matrix of the objective function [30]. Ding and Wang [6] further demonstrated that the L-BFGS algorithm indeed can find out the optimal solution for the flood control problem defined by (4) – a non-differential objective function at the minima, even though the L-BFGS has been well known as a minimization algorithm for smooth optimization problems. Yu et al. [42], Lewis and Overtun [22], and Skajaa [37] proved that the BFGS algorithm is robust and convergent for the non-smooth optimization problem due to a locally non-differential function as shown in (4). Recently, Steward et al. [38] have compared the LBFGS with a new minimization algorithm called the limited-memory bundle method (LMBM) algorithm specifically designed to address large-scale non-smooth minimization problems. They have found that the LMBM yields results better than the L-BFGS, despite some stability problems. The LMBM may be able to eventually overcome the mathematical drawback of the L-BFGS. Investigation of the LMBM in river flood control would be an interesting topic, and will be performed in the near future by the authors.

According to the physical meanings of the control variables and the limitations of their values known from control devices (floodgates) and engineering experience for the capacity of discharge at a spillway or a pump station for flood diversion, the identification of the values subject to bound constraints about the capacities of the devices needs to be considered. There are several kinds of nonlinear optimization methods available for constrained optimization, e.g., penalty, barrier, augmented Lagrangian methods, sequential quadratic programming (SQP), etc. [30]. Although some optimization methods (e.g. SQP) can solve more complicated constrained problems with equality and/or inequality constraints, there are still limited algorithms capable of solving a large-scale optimization problem as needed in the present study. L-BFGS-B is an extension of the limited memory algorithm L-BFGS, which is, at present, the only limited memory quasi-Newton algorithm capable of handling both large-scale optimization problems and bounds on the control variables. For further descriptions of the L-BFGS and the L-BFGS-B algorithms, one may refer to Liu and Nocedal [24] and Byrd et al. [1]. In order to preserve the physical meanings of the control variables in every iteration step, the L-BFGS-B procedure validated by Ding et al. [4] is thus adopted in this study. The L-BFGS-B algorithm minimizes the objective function \( J \) in Eq. (4) subject to the simple bound constraints, i.e. \( q_{\text{min}} \leq q \leq q_{\text{max}} \), where \( q_{\text{min}} \) and \( q_{\text{max}} \) mean, respectively, the maximum and minimum lateral outflow rates in a floodgate.

In optimization process for searching for the best flood diversion rate \( q(t) \), the searching process of the minimization procedure should start with a small initial value in order to avoid a big diversion discharge to suddenly suck flood flows out of channels to create an unrealistic and unfavorable supercritical flow in river channels. Mathematically speaking, even though a large withdrawal discharge which may be sufficient to dry up the channel bed could make the objective function (4) equal to zero, it is not a physical solution for realistic river flood control. Another advantage for searching the diversion discharge from a small value is to guarantee that the optimal withdrawal hydrograph minimizes the objective function (4) to protect the target river reaches from being overflown, and to assure that the least flood waters are diverted from the river channel.

6. Numerical examples for optimal flood diversion control

As shown in Fig. 3, a dendritic channel network of a watershed consisting of a main stem, two branches, and two confluences is used for testing this optimal flood control model. For simplicity, a compound cross-section is assumed as the channel shape for all three channels, even though the CCHE1D flood model and the adjoint model are applicable to a natural river reach with an irregular cross section. Three triangular hydrographs are imposed respectively at the three upstream inlets of the channels. The parameters to define the triangular hydrographs (\( Q_p, Q_0 = \) peak and base flow discharges, respectively; \( T_p = \) duration of storm or period of control action) are listed in Table 1. The hydrograph at the inlet of Channel 3 (main stem) has a slightly higher peak discharge (60 m³/s) than the other two. As shown in the table, three different values of Manning’s \( n \) are specified on the three channels to represent the different bed roughnesses. For the simulation of flood propagation, the channels are divided into a total of 43 short reaches (i.e. 48 cross sections) with equal spatial increment (500 m). The control duration for mitigating the flood water by diverting channel flow through the floodgate is 50 h, which covers the whole storm period. The time step \( \Delta t \) for simulating channel flows and the adjoint variables is 5 min. The weight factor \( W_i \) is set to 1000.

6.1. Verification of adjoint sensitivity

Based on the above adjoint sensitivity analysis (ASA), the sensitivity of the objective function with respect to the control action, i.e. the flow diversion rate \( q(t) \), is determined by Eq. (12), namely,
the adjoint variable \( \lambda_A \) at the locations of floodgates. Due to nonlinear nature of the adjoint equations (8) and (9), there are no directly analytical solutions for the two adjoint variables. Therefore, in general, sensitivity of the function cannot be verified by comparing computed values with any analytical solutions of the adjoint equations. However, utilizing a difference quotient as an approximation of the gradient of the function with respect to the control variable, the direct sensitivity method (DSM) can be used to estimate sensitivities of the function. Yeh [40] and Yeh and Sun [41] provided a mathematical definition of the direct sensitivity and a procedure for calculating the sensitivities by using a difference scheme. A difference approximation of the sensitivity of \( J \) with respect to \( q \) can be written as follows:

\[
\frac{\partial J}{\partial q} = \frac{J(q_0(x_c, t) + \delta q(x_c, t)) - J(q_0(x_c, t))}{\delta q(x_c, t)}, \tag{41}
\]

where \( q_0 \) = an initial diversion rate, and \( \delta q(x_c, t) \) = a small amount of perturbed diversion rate at a time \( t_0 \) and the location of floodgate. Sanders and Katopodes [36] have applied the DSM to verify adjoint sensitivities of a quadratic objective function for open channel flow control. They produced good estimations of the sensitivities by the DSM in comparison with the results obtained by the ASA. Before introducing the DSM for verification of the present adjoint sensitivity, it has to be noticed that due to the non-smooth (or non-differential) nature of the objective function (4), the DSM cannot be directly applied to estimate the sensitivity as Sanders and Katopodes [36] have done. But if we specify sufficient lower values for the allowable water stages \( Z_{obj} \) at target reaches \( x_0 \), the calculation of the objective function (4) can rule out the chance of checking the inequality in this non-smooth function, the function turns out to be differential in all the range of water elevations in the target reaches. Then, the DSM can be used for verification of the adjoint sensitivity under this special condition of the control water stage. Consequently, three test cases for estimation of the sensitivities with respect to a single floodgate diversion in the watershed were performed. This floodgate is located at the downstream and indicated by an arrow in Fig. 3. Each test case controlled only one river reach at a time with an equal length of 500 m. The three target reaches are St.1, St.2, and St.6 shown in Fig. 3. In the direct sensitivity analysis using Eq. (41), a zero allowable stage, i.e. \( Z_{obj}(x_0) = 0 \) m, is specified to three target reaches respectively to assure that all the variations of the water stages due to the perturbations will be counted in the objective function. Having used different values for the perturbed diversion rate \( \delta q \) at the range from 0.00001 to 10.0, we finally found convergent sensitivities at \( \delta q = 0.5 \) and 1.0. In addition, the adjoint variables are computed by using the initial diversion rate \( q_0(x_c, t) \) equal to 0.0. Fig. 4 presents the intercomparison of the sensitivities computed by the adjoint variable \( \lambda_A \) and the DSM in the three cases with different target reaches. It implies that the results by the DSM are very close to those by the ASA, or in other words, the adjoint sensitivities defined by the variable \( \lambda_A \) are consistent with the direct sensitivities estimated by the difference quotient (41). Even though the DSM can give good estimations of the sensitivities of the function, due to the tedious perturbations at every time step (showing by the symbols in Fig. 4) to calculate individual sensitivity value by (41), the DSM is very time-consuming.

### 6.2. Optimal flood diversion with single floodgate

In this case, by using the ASA, only one floodgate for flood diversion control is assumed to be placed at the downstream of the main stem, as shown in Fig. 3. To protect all the three channels from being overflowed, the maximum allowable stage \( Z_{obj} \) is set to be 3.5 m over the entire channel network. The optimization of the flood control problem turns out to find the optimal flood diversion hydrograph \( q(t) \) by minimizing the overflow water stages over the 50-h optimization duration and the entire watershed. The searching process started with a small initial value of \( q(t) \), which was 0.001 m³/s. It assures that the searching direction is from a

---

**Table 1**

<table>
<thead>
<tr>
<th>Channel no.</th>
<th>( Q_p ) (m³/s)</th>
<th>( Q_b ) (m³/s)</th>
<th>( T_p ) (h)</th>
<th>( T ) (h)</th>
<th>( Z_{obj} ) (m)</th>
<th>( n ) (s/m³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>50.0</td>
<td>2.0</td>
<td>16.0</td>
<td>48.0</td>
<td>3.5</td>
<td>0.01</td>
</tr>
<tr>
<td>2</td>
<td>50.0</td>
<td>2.0</td>
<td>16.0</td>
<td>48.0</td>
<td>3.5</td>
<td>0.02</td>
</tr>
<tr>
<td>3</td>
<td>60.0</td>
<td>2.0</td>
<td>16.0</td>
<td>48.0</td>
<td>3.5</td>
<td>0.03</td>
</tr>
</tbody>
</table>

**Fig. 3.** Configuration of a channel network with a floodgate.
small diversion rate to a large rate in order to avoid unphysical big diversion discharge to dry up the channel bed. As shown in Fig. 5, the iterations for searching for the optimal lateral outflow \( q(t) \) by the L-BFGS-B algorithm indicate that after less than 30 iterations, the convergent optimal solution of the \( q(t) \) was found. Here, the L-BFGS-B with the bound constraint optimization (i.e. \( q(t) < 0 \)) prevented the lateral outflows from recharging the channel (i.e. \( q(t) \)).
flow from outside toward the channel). The identified optimal hydrograph $q(t)$ to schedule the flood water diversion shows that the floodgate should discharge 10 m$^3$/s flood water at the beginning of the storm; then, the peak discharge of the lateral outflow reaches up to 130.0 m$^3$/s at 17.5 h, which is a 1.5-h delay by comparing with the arrival time of the peak discharge at the upstream.

Notice that in the optimization the lateral outflow variable $q(t)$ is a vector with the number of components equal to the number of

---

**Fig. 8.** Temporal profiles of the two adjoint variables at five monitoring stations at the first iteration (i.e. no lateral outflow diverted from the floodgate).
total time steps over the simulation period. Here, the control vector $q(t)$ contains 601 components at the flood duration = 50 h, and the time step = 5 min. That means that 601 values of the parameter $q(t)$ have to be identified by the L-BFGS-B optimization algorithm. For explaining the performance of the optimization procedure, the optimal control model was run iteratively till the values of the control parameters was no longer changed (i.e. as small as a machine precision). In the meantime, the double precision arithmetic for solving the adjoint variables is adopted. Fig. 6 shows the iterative histories of the objective function values and the norm.

Fig. 9. Comparisons of stages between controlled and uncontrolled states at five stations.

Fig. 10. Comparisons of discharges between controlled and uncontrolled states at five stations.
of its gradient (i.e. $\nabla J(q)L$). It is found that starting with a small guess value of the diversion rate, the objective function value decreases monotonically and the control variable converges after 15 iterations using L-BFGS-B. The discharge obtained after 15 iterations is considered as the optimal solution, because the gradient is very close to zero and the objective function are therefore minimized. Moreover, the flood control model took less than one second of CPU time for executing one iteration of the optimization in a PC with 2.66 GHz CPU. For this kind of optimization problem with 48 cross sections in the channel network and 601 control parameters, it only took 11 s of CPU time for the integrated simulation-based control model to find the optimal solution at the 15th L-BFGS-B iteration.

In fact, optimization efficiency depends on the accuracy of the adjoint sensitivity of the objective function, which is determined by the adjoint variable $\lambda(t)$ at the control location as shown in Eq. (12). Fig. 7 presents the iteration process of the adjoint variable $\lambda(t)$ at the location of the floodgate as L-BFGS-B is searching for the optimal solution. At the beginning of iterations, the parabolic curves of the adjoint variable reflect the temporal distribution of the discrepancy between the predicted and the allowable water stages. It indicates that the lateral outflow rate should increase along with the flood hydrograph. After a few L-BFGS-B iterations, the peak of the adjoint variables gradually shift toward the flood peak at $t = 16$ h. At the 10th iteration, the values of $\lambda(t)$ become very small and close to the machine precision, and then almost reach to zero at the 20th iteration. It means that the obtained solution of the outflow hydrograph in Fig. 6 is indeed mathematically optimal.

To look into the temporal/spatial distributions of the two adjoint variables, five monitoring stations whose locations are shown in Fig. 3 are chosen to visualize the variations of the variables in the main stem and the two branches in the watershed. Temporal profiles of the two adjoint variables at the first iteration (i.e. no lateral outflow diverted from the floodgate) are plotted in Fig. 8. The variations of $\lambda(t)$ at the upstream of the main stem and the two branches (i.e. St. 1, St.2, St. 4, and St. 5) are much larger than that at downstream (e.g. St. 3). It implies that $\lambda(t)$ is more sensitive at upstream than that at downstream, and therefore the diversion control located at upstream should be more effective than downstream. In contrast, the changes of $\lambda(t)$ at downstream are greater than those at upstream due to the boundary condition of the adjoint variable at inlets, which is given by Eq. (16).

As depicted in Fig. 9, the temporal variations of the controlled water stages over the 50-h storm period at the five selected stations are compared with those stages without diversion control. The flood control results show that the stages at almost all the stations through the flood period are lower than the allowable stage (i.e. 3.5 m); the flood waters in the main channel and two branches over the entire storm period, therefore, are well-controlled. There is only a very short period when the water stages at upstream (St. 1 and St. 4) are slightly higher than the allowable stage. Meanwhile, the flood water at the downstream (St. 3) in the main channel forms two small water surface waves.

Further comparisons of discharges between controlled and uncontrolled states at the five monitoring stations are shown in Fig. 10. Due to the flood diversion, the peak discharges at the downstream of the floodgate (i.e. St. 3) is reduced significantly. However, remarkable increases of the discharges occurred at the upstream stations, St. 2. One possible reason for the discharge increasing at this station is that the control constraint of the allowable maximum stage (3.5 m) over the entire watershed is so restrictive so that before the peak flood arrives, the floodgate has to divert in advance a large amount of flood waters out of the channel. Consequently, the flow velocities at upstream become faster than those without flood water withdrawal, and the speed-up channel flows at upstream may cause unexpected erosion on river bed in the case of controlling real watershed with a movable bed. To avoid this problem, one may relax the constraint of the stages at upstream by constructing dikes following
a longitudinal slope of the river flow, by which the ability of flood prevention at upstream will be strengthened. Another way is to balance the load of the lateral outflow by diverting flood waters from multiple floodgates, which is discussed in the following control case.

6.3. Optimal flood diversion with multiple floodgates

Installation of multiple floodgates in river banks of different river reaches is a common practice to sequentially divert flood waters from upstream to downstream in a storm period. As a rule
of thumb, flood control by operating more than floodgate in a river basin can better balance the pressure of flood waters in all river reaches due to unsteadiness and non-uniformities of flood flows. In this case, it is assumed that there are three floodgates equally spaced in the main stem of the channel network, as shown in Fig. 11, which is the same watershed as the previous case. The constraint for preventing flood water overflowing is set as the same as the previous case with one floodgate diversion, i.e. 3.5-m maximum allowable water elevations for the entire channel network.

By using L-BFGS-B, three vectors of lateral outflow hydrographs, i.e. \( q_1(t) = q(x_1, t) \), \( q_2(t) = q(x_2, t) \), and \( q_3(t) = q(x_3, t) \), consisting of a totaling 1803 components (601 time steps for each outflow hydrograph) have to be identified. By this simulation-based flood control model, the optimal solutions of the three diversion hydrographs at the three floodgates were obtained as shown in Fig. 12. Notice that the peak diversion discharge at the first floodgate is about 50 m\(^3\)/s, which is much less than the peak discharges value (130 m\(^3\)/s) by the single floodgate diversion in the previous case. It indicates that more waters need to be diverted at upstream than

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**Fig. 15.** Comparison of objective functions between one and three floodgates control.

**Fig. 16.** A reach of the East Fork River digitized from a DEM in the CCHE1D GUI. Note that the river flows from the south toward the north. The points represent the locations of the cross sections for the computation.

**Fig. 17.** Observed hydrograph at the inlet cross in a period from May 20 to June 19, 1979.
those at downstream, and the individual hydrograph in the three gates is much less than the one withdrawn by the single floodgate in the previous case. Moreover, the total discharge, i.e. $q_1(t) + q_2(t) + q_3(t)$, withdrawn by the three gates, is even less than the one by the single gate operation. It denotes that multiple floodgate control is more effective than that by a single gate, and can be operated easily, because the control load is distributed along the channel. By considering the total volume 13.824 million m$^3$ of the flood waters coming from the inlet over the 48-h flood, the percentages of the diverted flood waters by the single floodgate and the three floodgates are compared in Table 2. It shows that the single floodgate control operation needs to divert 60.8% of the floodwaters, but the three floodgates only divert totally 47%. Apparently, multiple floodgate operation is a cost-effective flood control approach if the optimal control can be achieved. The results for the maximum diversion discharges and the volume of the diverted flood waters can be further used as the basic data to design the minimum capacities of the floodgates and the flood water detention basins.

As shown in Fig. 13, controlled by the three floodgates, all the water stages at the five monitoring stations, of which the locations are shown in Fig. 3, didn’t exceed the allowable stage (3.5 m). The controlled stages in the main stem (St. 1, St. 2, and St. 3) and the two branches (St. 4 and St. 5) are more stable (flatter) than these by the single floodgate control as shown in Fig. 9. As shown in Fig. 14, the controlled discharges in the main stem are much smaller than these without control, though the controlled discharges in the two branches (St. 4 and St. 5) slightly increased. It means that the multiple floodgate control can avoid channel flow speeding up by over-withdrawing flood waters from the main stem by a single floodgate as found in the previous case, by which may cause extra river bed erosion.

To compare the optimization performance of the two control actions, Fig. 15 plots the values of the objective function varying in the searching processes in the single and multiple floodgate control actions. It shows that the optimization algorithm (L-BFGS-B) can quickly find (in about 20 iterations) the optimal flood diversion discharge hydrographs, no matter the floodgate is single or multiple. However, the final value of the objective function by the three floodgate control is four digits smaller than that by the single floodgate. It means that the multiple floodgate control can achieve the global optimal control in the entire watershed easier than the single-gate control. It also reveals that this optimization methodology and the simulation-based flood control tool are capable of determining the number, locations, capacities of the floodgates in rivers and watersheds.

6.4. Optimal flood diversion in a natural river

In order to examine the capability of this simulation-based optimization for controlling flood flows in a real river with a naturally complex geometry, a river reach in the East Fork River in western Wyoming [8], as shown in Fig. 16, was selected. This study reach is 3.3-km long, and divided into 41 cross-sections with unequal spatial length. The flow control in this river is demonstrated by controlling a flow diversion during a real flood observed in a one-month period from May 20 to June 19, 1979. The observed hydrograph at the inlet shown in Fig. 17 contains three flood peaks, in which the maximum peak discharge is about 33 m$^3$/s. A hypothetical floodgate is installed at Section No. 5 close to the upstream (Fig. 16). The control target water stages or the allowable water elevations $Z_{obj}(x)$ vary along the reach, as shown in Fig. 18, which form a slope line between the minimum and maximum bank elevations. This simulation-based optimization software was used for searching for the best diversion schedule (hydrograph) at the hypothetical floodgate in order to control the flood water stages...
being lower than the allowable elevations. The time step for the flow simulations is 15 min. Obtained by a previous model calibration study using observations in the river [4], the bed roughness coefficients (i.e. Manning’s $n$) varying along the reach were used for the river flow simulations. The downstream boundary condition at the outlet was given by a discharge-stage rating curve.

Fig. 19 depicts the searching process (or the history of the objective function values measured by Eq. (4)) of the optimal flow diversion control using the L-BFGS-B algorithm. It indicates that only through 20 L-BFGS-B iterations the value of the objective function reaches its minimum. The computational CPU time for performing all the 50 searching iterations was only a few minutes on a PC. The obtained optimal flood water diversion hydrograph is presented in Fig. 20. The optimal diversion hydrograph implies that the diversion in the first flood wave plays a major role in the 30-day flood control operation. Because the last two flood waves

Fig. 21. Comparisons of water stages between the optimal control and no control. Note that the red lines represent the allowable water elevation $Z^{\text{obj}}$ at the corresponding sections. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Fig. 22. Comparisons of discharges at three cross sections between the optimal control and no control.
are relatively small, the diversion operation only is to cut a small amount of the peak discharge (about 5 m$^3$/s) in the two last flood flows. In Fig. 21, the water elevations at three cross sections with the optimal diversion control are compared with those with no control. All the stages with the optimal flow control are lower than the variable target elevations Z$^{opt}$(x), which are given in Fig. 18. Moreover, Fig. 22 gives the comparisons of the discharges at the three sections between the optimal diversion and no control. The controlled discharge at the crossection of the floodgate clearly shows that the diversion flows cut the first flood peak, and end up with an almost constant value of discharge passing onto downstream. The discharges in the two last floods also indicate that both two small flood flows won’t cause too much flooding at the downstream; most of the river flows after the first peak can pass through this river reach.

7. Conclusions

In this study, nonlinear numerical optimization approach based on the variational analysis for a flow simulation model is developed to determine the optimal flood diversion control operation to mitigate flood water stages in channel network of a watershed. This integrated simulation-based optimization model for control flood stages in watershed consists of a well-established 1-D flow model called CCHE1D and its adjoint model. The adjoint equations of the dynamic wave model in CCHE1D are derived based on the variational principle. The sensitivity of the objective function with respect to the control variables, i.e. flood diversion discharges at floodgates, is computed by the adjoint sensitivity represented by two adjoint variables. To solve the two adjoint equations in the watershed, the internal conditions on the confluences in channel network are derived accordingly. The implicit Preissmann’s schemes are implemented for solving both the flow governing equations and the adjoint equations in channel network. Both the flood simulation model and the adjoint model are efficient and robust.

By incorporating into this 1-D flow simulation application software, this optimal control model enables to solve the practical flow control problems in flood diversion control in rivers and channel network of a watershed. Two examples for testing flood diversion control are demonstrated for mitigating flood water stages in channel network by finding the optimal flood diversion discharges at floodgates. The performance of different floodgate installations in watershed is investigated by testing the effectiveness of the diversion control through a single floodgate and multiple floodgates. It is found that the multiple floodgate control is a cost-effective approach to better balance the diversion flood waters over the entire watershed, and it can also avoid river flows speeding up by withdrawal of waters as found in the single floodgate control. Overwithdrawing flood waters may create unexpected erosion on river bed around flood diversion facility. Furthermore, a one-month optimal flood control was achieved by applying the CCHE1D-based optimization to mitigate river floods in a natural river with complex river course and cross-section shapes. Numerical optimization shows that the developed optimization model can quickly find the optimal solutions in all the cases with various control conditions. Collaborating with CCHE1D, this optimal control model becomes generally applicable for practicing the optimal control of flood diversion in a river and a channel network of a watershed. By evaluating the control action performance, it is demonstrated that this optimization methodology and this integrated optimization software are capable of determining the best schedules of the optimal diversion discharge, the optimal locations, the minimum capacities of the floodgates and detention basins in river and watershed. Therefore they can assist flood water managers and decision makers to seek the best planning, management and design for flood diversion control in rivers and watersheds.

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